Dualité de Koszul et algèbres de Lie semi-simples en caractéristique positive

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Dualité de Koszul et algèbres de Lie semi-simples en caractéristique positive

Koszul duality and semi-simple Lie algebras in positive characteristic

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Introduction

Depuis les travaux de Beilinson, Ginzburg et Soergel (voir [BGS96]), la dualité de Koszul et la notion d'anneau de Koszul sont devenues des ingrédients essentiels en théorie de Lie (voir par exemple [AJS94]). L'un des résultats principaux de la présente thèse est la construction d'une "dualité de Koszul géométrique" reliant différentes catégories dérivées de représentations de l'algèbre de Lie $\mathfrak{g}$ d'un groupe algébrique semi-simple, connexe, simplement connexe sur un corps algébriquement clos de caractéristique positive p, et l'étude de ses propriétés. En particulier nous en déduisons que l'algèbre enveloppante restreinte de $\mathfrak{g}$ peut être munie d'une graduation de Koszul si p est suffisamment grand, et nous donnons des informations sur son anneau dual.

1 Contexte

1.1 Anneaux de Koszul

La notion d'anneau de Koszul a été définie en 1970 par Priddy dans [Pri70]. Sauf mention explicite, les anneaux considérés ne seront jamais supposés commutatifs. Un anneau gradué

$$A = \bigoplus_{i \in \mathbb{Z}} A_i$$

est dit *de Koszul* s'il vérifie les propriétés suivantes :

1. $A_i = 0$ si $i < 0$;
2. A_0 est un anneau semi-simple ;
3. Le A-module $A_0 \cong A/A_{>0}$ admet une résolution projective graduée

   $$\cdots \to P^2 \to P^1 \to P^0 \to A_0 \to 0$$

   telle que, pour tout $i \geq 0$, P^i est engendré sur A par sa partie de degré i.

Les conditions 1 et 2 sont faciles à comprendre. Si elles sont vérifiées, en termes plus concrets, la condition 3 implique que l'anneau A est engendré (comme A_0-algèbre) par des éléments de degré 1, que les relations entre ces générateurs sont engendrées en degré 2, que les relations entre ces relations sont engendrées en degré 3, et ainsi de suite (les relations d'ordre n sont engendrées en degré $n + 1$).

L'exemple le plus simple d'anneau de Koszul (hors les anneaux semi-simples, concentrés en degré 0) est celui de l'algèbre symétrique $\mathrm{S}(V)$ d'un espace vectoriel V de dimension finie, placé en degré 1.

Si A est un anneau gradué en degrés positifs ou nuls, on peut considérer l'anneau gradué

$$E(A) \ := \ \bigoplus_{n \geq 0} \mathrm{Ext}^n_A(A_0, A_0),$$

muni du produit de Yoneda, où les groupes Ext^n_A sont pris dans la catégorie des A-modules (non gradués). Si A est un anneau de Koszul, tel que A_1 est un A_0-module de type fini, alors l'anneau $A^! := E(A)^{\mathrm{op}}$ est également un anneau de Koszul, appelé *anneau dual*. Notons que sous ces hypothèses on a un isomorphisme canonique $(A^!)^! \cong A$.

Par exemple, l'anneau dual de l'anneau de Koszul $\mathrm{S}(V)$ est l'algèbre extérieure $\Lambda(V^*)$ du dual V^* de V. Ici encore, V^* est placé en degré 1.

1.2 Les travaux de [BGS96]

Dans l'article [BGS96], les auteurs démontrent que chaque bloc de la catégorie $\mathcal{O}$ d'une algèbre de Lie semi-simple complexe $\mathfrak{g}_{\mathbb{C}}$ est "gouverné" par un anneau de Koszul, c'est-à-dire est équivalent à la catégorie des modules non-gradués de type fini sur un anneau de Koszul. Dans le cas d'un bloc régulier, cet anneau est auto-dual, c'est-à-dire isomorphe à son anneau dual. Dans le cas d'un bloc singulier, la catégorie des modules de type fini (non gradués) sur l'anneau dual est également équivalente à une sous-catégorie explicite de la catégorie $\mathcal{O}$.

La preuve de ce résultat est basée sur une description géométrique des blocs de la catégorie $\mathcal{O}$. Plus précisément, la théorie de la localisation due à Beilinson et Bernstein donne des équivalences de catégories entre certaines catégories (abéliennes) de $\mathfrak{g}_{\mathbb{C}}$-modules et certaines catégories de $\mathcal{D}$-modules sur la variété des drapeaux $\mathcal{B}_{\mathbb{C}}$ associée à $\mathfrak{g}_{\mathbb{C}}$. Il s'agit de la partie "algébrique" de la description. Ensuite vient une partie "topologique" : la correspondance de Riemann-Hilbert identifie ces catégories de $\mathcal{D}$-modules à certaines catégories de faisceaux pervers sur $\mathcal{B}_{\mathbb{C}}$. Schématiquement, on a donc la description suivante :

$$\mathfrak{g}_{\mathbb{C}}\text{-modules} \xrightarrow[\text{(Algèbre)}]{\text{Localisation}} \left\{ \begin{array}{c} \mathcal{D}\text{-modules} \\ \text{sur } \mathcal{B}_{\mathbb{C}} \end{array} \right\} \xrightarrow[\text{(Topologie)}]{\text{Riemann-Hilbert}} \left\{ \begin{array}{c} \text{Faisceaux} \\ \text{pervers sur } \mathcal{B}_{\mathbb{C}} \end{array} \right\}.$$

1.3 Localisation en caractéristique positive

Dans les articles [BMR08] et [BMR06], Bezrukavnikov, Mirković et Rumynin ont développé un analogue de la description géométrique précédente en caractéristique positive. Plus précisément, considérons l'algèbre de Lie $\mathfrak{g}$ d'un groupe algébrique semi-simple, connexe, simplement connexe G sur un corps algébriquement clos de caractéristique p. Dans cette sous-partie, nous supposerons que p est supérieur au nombre de Coxeter h de G.

La première étape (algébrique) de leur construction consiste à démontrer que les analogues des foncteurs considérés par Beilinson et Bernstein induisent des équivalences de catégories *dérivées* entre certaines catégories de $\mathfrak{g}$-modules et certaines catégories de $\mathcal{D}$-modules (cristallins) sur la variété des drapeaux $\mathcal{B}$ associée à G.

La correspondance de Riemann-Hilbert n'admet pas d'analogue en caractéristique positive dans ce contexte. En remplacement, les auteurs de [BMR08] utilisent des arguments géométriques : la propriété d'Azumaya du faisceau d'algèbres d'opérateurs différentiels sur une variété lisse en caractéristique positive permet de démontrer des équivalences de catégories (abéliennes) entre les catégories de $\mathcal{D}$-modules considérées et certaines catégories de faisceaux cohérents sur la variété $\widetilde{\mathfrak{g}}^{(1)}$, où l'exposant ${}^{(1)}$ désigne le décalage de Frobenius, et où $\widetilde{\mathfrak{g}}$ est la "résolution simultanée" de Grothendieck (un certain fibré vectoriel au-dessus de $\mathcal{B}$). Schématiquement, on a donc la description suivante :

$$\mathfrak{g}\text{-modules} \xrightarrow[\text{(Algèbre)}]{\text{Localisation dérivée}} \left\{ \begin{array}{c} \mathcal{D}\text{-modules} \\ \text{sur } \mathcal{B} \end{array} \right\} \xrightarrow[\text{(Géométrie)}]{\text{Azumaya}} \left\{ \begin{array}{c} \text{Faisceaux} \\ \text{cohérents sur } \widetilde{\mathfrak{g}}^{(1)} \end{array} \right\}.$$

Nous renviendrons plus en détail sur cette théorie en 2.1 ci-dessous.

1.4 Koszulité de l'algèbre enveloppante restreinte

Gardons les notations de la sous-partie 1.3, et notons $(\mathcal{U}\mathfrak{g})_0$ l'algèbre enveloppante restreinte de $\mathfrak{g}$. Dans l'article [AJS94], Andersen, Jantzen et Soergel démontrent que, pour p suffisamment grand (sans borne explicite), les blocs réguliers de $(\mathcal{U}\mathfrak{g})_0$ peuvent être munis d'une graduation de Koszul.

Dans le chapitre III de cette thèse (voir 2.3 ci-dessous) nous obtenons en particulier une nouvelle preuve de ce résultat comme corollaire de nos constructions. Nous donnons également des informations sur l'anneau de Koszul dual, et nous étendons cette propriété aux blocs singuliers.

2 Présentation des résultats

Ce mémoire se compose de quatre chapitres.

2.1 Rappels et calculs explicites

Dans le chapitre I, nous rappelons tout d'abord les résultats principaux des articles [BMR08] et [BMR06]. Supposons comme ci-dessus que la caractéristique p est supérieure au nombre de Coxeter de G. Soit $T \subset G$ un tore maximal, et $\mathfrak{t}$ son algèbre de Lie. Le centre $\mathfrak{Z}$ de l'algèbre enveloppante $\mathcal{U}\mathfrak{g}$ de $\mathfrak{g}$ est engendré par deux sous-algèbres : le *centre de Harish-Chandra* $\mathfrak{Z}_{\mathrm{HC}}$ et le *centre de Frobenius* $\mathfrak{Z}_{\mathrm{Fr}}$. Un caractère de $\mathfrak{Z}$ est donc donné par une "paire compatible" (λ, χ) où $\lambda \in \mathfrak{t}^*$ (alors λ définit un caractère de $\mathfrak{Z}_{\mathrm{HC}}$) et $\chi \in \mathfrak{g}^*$ (alors χ définit un caractère de $\mathfrak{Z}_{\mathrm{Fr}}$).

Dans ce mémoire nous considérons uniquement[1] le cas où λ est l'image d'un caractère de T (que l'on note également λ) et où χ est nilpotent (et même $\chi = 0$ la plupart du

[1] Notons qu'on peut toujours se ramener à ce cas si l'on accepte de considérer des groupes réductifs plutôt que semi-simples, voir par exemple [Jan98, 7.4].

temps). On note $(\mathcal{U}\mathfrak{g})^{\lambda} := (\mathcal{U}\mathfrak{g}) \otimes_{\mathfrak{Z}_{\mathrm{HC}}} \Bbbk_{\lambda}$, $(\mathcal{U}\mathfrak{g})_{\chi} := (\mathcal{U}\mathfrak{g}) \otimes_{\mathfrak{Z}_{\mathrm{Fr}}} \Bbbk_{\chi}$ les algèbres obtenues par spécialisation. En particulier, $(\mathcal{U}\mathfrak{g})_0$ est l'algèbre enveloppante restreinte considérée en 1.4. On note également $\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,\chi)}(\mathcal{U}\mathfrak{g})$ la catégorie des $\mathcal{U}\mathfrak{g}$-modules de type fini sur lesquels $\mathfrak{Z}$ agit avec un caractère *généralisé* (λ, χ), et on utilise des notations similaires pour les catégories $\mathrm{Mod}^{\mathrm{fg}}_{\chi}((\mathcal{U}\mathfrak{g})^{\lambda})$, $\mathrm{Mod}^{\mathrm{fg}}_{\lambda}((\mathcal{U}\mathfrak{g})_{\chi})$.

Comme en 1.3, soit $\widetilde{\mathfrak{g}}$ la résolution simultanée de Grothendieck, et soit $\widetilde{\mathcal{N}} \subset \widetilde{\mathfrak{g}}$ la variété de Springer. On considère $\mathcal{B}$ comme la section nulle de $\widetilde{\mathcal{N}}$ et $\widetilde{\mathfrak{g}}$. Alors si λ est un caractère régulier, la théorie de la localisation en caractéristique positive donne notamment des équivalences de catégories

$$\begin{array}{rrcl} \epsilon^{\mathcal{B}}_{\lambda} : & \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) & \xrightarrow{\sim} & \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_0((\mathcal{U}\mathfrak{g})^{\lambda}) \ ; \\ \gamma^{\mathcal{B}}_{\lambda} : & \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}) & \xrightarrow{\sim} & \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,0)}(\mathcal{U}\mathfrak{g}). \end{array}$$

Ici, pour Y un sous-schéma fermé d'un schéma X, on a noté $\mathrm{Coh}_Y(X)$ la catégorie des faisceaux cohérents sur X supportés (ensemblistement) dans Y.

À la suite de ces rappels nous présentons des calculs explicites dans les cas où $G = \mathrm{SL}(2)$ et $G = \mathrm{SL}(3)$ (obtenus en collaboration avec Roman Bezrukavnikov, et publiés dans un appendice à [BMR08]). Plus précisément, dans ces deux cas nous déterminons les images inverses par l'équivalence $\epsilon^{\mathcal{B}}_0$ des objets simples de la catégorie $\mathrm{Mod}^{\mathrm{fg}}_0((\mathcal{U}\mathfrak{g})^0)$. Nous calculons également des objets de la catégorie $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)})$ ayant le "comportement homologique" d'objets projectifs de la catégorie $\mathrm{Mod}^{\mathrm{fg}}_0((\mathcal{U}\mathfrak{g})^0)$ via $\epsilon^{\mathcal{B}}_0$. Notons que la catégorie abélienne $\mathrm{Mod}^{\mathrm{fg}}_0((\mathcal{U}\mathfrak{g})^0)$ ne possède aucun object projectif.

Les calculs dans le cas de $G = \mathrm{SL}(2)$ sont faciles, mais ils seront utiles car ils se généralisent pour déterminer certains objets dans le cas général (nous présentons et utilisons cette généralisation dans le chapitre III). Dans le cas de $G = \mathrm{SL}(3)$, les calculs deviennent plus difficiles.

2.2 Action géométrique du groupe de tresses

Dans le chapitre II nous présentons une construction qui aura un rôle technique important dans le chapitre III, mais qui a également un intérêt propre.

Notons B'_{aff} le groupe de tresses affine étendu associé à G. Pour presque toute caractéristique p (et en particulier si $p = 0$) nous construisons par des méthodes géométriques une action[2] du groupe B'_{aff} sur la catégorie $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$. Le groupe B'_{aff} a deux types de générateurs : des éléments T_{s_α} associés aux réflexions simples du groupe de Weyl W de G (pour un certain choix d'une base du système de racines associé), et des éléments θ_x associés aux caractères de T. Pour cette action, l'élément θ_x agit par produit tensoriel avec le fibré en droites sur $\widetilde{\mathfrak{g}}$ associé naturellement à x.

Décrivons maintenant l'action des éléments T_{s_α}, dans le cas où la caractéristique p est très bonne (pour G). Dans ce cas, le groupe W agit de façon naturelle sur la restriction $\widetilde{\mathfrak{g}}_{\mathrm{rs}}$

[2]Ici nous considérons la notion faible d'action d'un groupe sur une catégorie : une action d'un groupe A sur une catégorie $\mathcal{C}$ est la donnée d'un morphisme de groupe de A vers le groupe des classes d'isomorphisme d'auto-équivalences de $\mathcal{C}$.

de $\widetilde{\mathfrak{g}}$ aux éléments réguliers semi-simples. Notons S_α l'adhérence dans $\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$ du graphe de s_α sur $\widetilde{\mathfrak{g}}_{\mathrm{rs}}$. Alors T_{s_α} agit par convolution (ou transformée de Fourier-Mukai) avec le noyau $\mathcal{O}_{S_\alpha}$.

Cette action se factorise également en une action de B'_{aff} sur la catégorie $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$.

Nous présentons deux preuves du fait que ces foncteurs donnent lieu à une action de B'_{aff}. La première démontre le résultat dans le cas où G n'a pas de composante de type $\mathbf{G}_2$ et $p \neq 2$ si G a une composante de type $\mathbf{B}$, $\mathbf{C}$ ou $\mathbf{F}$. Elle a été publiée dans [Ric08a]. La seconde preuve est valide pour tout groupe G, si la caractéristique p est très bonne[3]. Il s'agit d'un travail en collaboration avec Roman Bezrukavnikov.

Cette action a plusieurs interprétations en théorie des représentations. Supposons tout d'abord que la caractéristique p est positive, et supérieure au nombre de Coxeter de G. Dans ce cas, Bezrukavnikov, Mirković et Rumynin ont construit dans [BMR06] une action du groupe de tresses B'_{aff} sur chacune des catégories $\mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,\chi)}(\mathcal{U}\mathfrak{g})$ pour λ régulier et χ nilpotent. Via les équivalences $\gamma^{\mathcal{B}}_\lambda$ et leurs analogues pour $\chi \neq 0$, elles induisent des actions de B'_{aff} sur diverses sous-catégories de $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})$. Nous démontrons que la décalée par le Frobenius de l'action considérée ci-dessus sur $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$ se restreint en les actions de [BMR06] sur toutes ces sous-catégories.

Supposons maintenant que le corps de base est $\mathbb{C}$, et considérons l'action de B'_{aff} sur $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$. Comme pour $\widetilde{\mathfrak{g}}$, les générateurs θ_x et T_{s_α} agissent par convolution, et les noyaux associés sont des images directes de faisceaux sur le produit fibré $Z := \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$, une sous-variété fermée de $\widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}$ (appelée *variété de Steinberg*). Ici $\mathcal{N}$ est la variété nilpotente de $\mathfrak{g}$, et le morphisme $\widetilde{\mathcal{N}} \to \mathcal{N}$ est la résolution de Springer. D'autre part, ces noyaux sont naturellement $G \times \mathbb{C}^\times$-équivariants, où $\mathbb{C}^\times$ agit sur Z par dilatation dans les fibres de la projection $Z \to \mathcal{B} \times \mathcal{B}$. L'action est donc définie par un morphisme de groupes de B'_{aff} vers le groupe des classes d'isomorphismes d'objets de la catégorie $\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{C}^\times}_Z(\widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}})$ (muni du produit de convolution). Passant à la K-théorie, on obtient un morphisme de groupes de B'_{aff} vers la K-théorie $G \times \mathbb{C}^\times$-équivariante de Z, qui est isomorphe (d'après Ginzburg et Kazhdan-Lusztig, voir [CG97] ou [Lus98]) à l'algèbre de Hecke affine étendue $\mathcal{H}'_{\mathrm{aff}}$ associée à G. Cette algèbre est un certain quotient de l'algèbre de groupe de B'_{aff} sur $\mathbb{Z}[v, v^{-1}]$. Nous démontrons que le morphisme $B'_{\mathrm{aff}} \to \mathcal{H}'_{\mathrm{aff}}$ obtenu par cette construction est le morphisme naturel. Cette action est donc une *catégorification* de l'isomorphisme $K^{G\times\mathbb{C}^\times}(Z) \cong \mathcal{H}'_{\mathrm{aff}}$.

Enfin, toujours dans le cas du corps $\mathbb{C}$, cette action est liée à la construction géométrique (due à Springer) des représentations du groupe W dans la cohomologie des fibres de Springer.

2.3 Dualité de Koszul et $\mathcal{U}\mathfrak{g}$-modules

Le chapitre III présente les résultats principaux de cette thèse. Nous construisons une "dualité de Koszul" qui relie, pour λ un caractère régulier de T, les catégories dérivées

[3]Rappelons que cette hypothèse exclut les groupes ayant une composante de type autre que $\mathbf{A}$ si $p = 2$, de type $\mathbf{E}$, $\mathbf{F}$ ou $\mathbf{G}$ si $p = 3$, de type $\mathbf{G}$ si $p = 5$, ou de type $\mathbf{A}_{n-1}$ si p divise n.

$\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^\lambda)$ et $\mathcal{D}^b\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$, et montrons que cette dualité envoie les modules simples sur les modules projectifs indécomposables. Nous utilisons ensuite cette propriété pour reprouver, préciser et étendre certains résultats de [AJS94].

Cette approche fournit également des informations sur les images inverses de certains $\mathcal{U}\mathfrak{g}$-modules simples ou projectifs par les équivalences $\epsilon_\lambda^{\mathcal{B}}$ et $\gamma_\lambda^{\mathcal{B}}$ de 2.1.

Supposons que la caractéristique p est supérieure au nombre de Coxeter de G, et soit λ un caractère dans l'alcôve fondamentale. Via l'équivalence $\epsilon_\lambda^{\mathcal{B}}$ de 2.1, la catégorie $\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^\lambda)$ s'identifie à une sous-catégorie de la catégorie $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)})$. En utilisant des arguments géométriques, nous construisons :

(a) des catégories triangulées "graduées" $\mathcal{C}^{\mathrm{gr}}$ et $\mathcal{D}^{\mathrm{gr}}$ (c'est-à-dire, munies d'une auto-équivalence notée $\langle 1\rangle$), qui sont des "versions graduées" des catégories $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)})$ et $\mathcal{D}^b\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$ respectivement (c'est-à-dire, on a des foncteurs "d'oubli de la graduation", $\mathrm{For} : \mathcal{C}^{\mathrm{gr}} \to \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)})$ et $\mathrm{For} : \mathcal{D}^{\mathrm{gr}} \to \mathcal{D}^b\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$;

(b) une équivalence $\kappa : \mathcal{C}^{\mathrm{gr}} \xrightarrow{\sim} \mathcal{D}^{\mathrm{gr}}$.

On obtient donc un diagramme

$$\begin{tikzcd}
 & \mathcal{C}^{\mathrm{gr}} \arrow[r, "\kappa", "\sim"'] \arrow[d, "\mathrm{For}"'] & \mathcal{D}^{\mathrm{gr}} \arrow[d, "\mathrm{For}"] \\
\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^\lambda) \arrow[r, hook] & \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)}) & \mathcal{D}^b\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0).
\end{tikzcd}$$

Notre résultat essentiel est alors que, pour $p \gg 0$, l'équivalence κ envoie les relevés des objets simples de $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^\lambda)$ sur les relevés des objets projectifs de $\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$, à un décalage près. Notons que la catégorie $\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$ est équivalente à la catégorie des modules sur un certain quotient $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$ de l'algèbre enveloppante restreinte $(\mathcal{U}\mathfrak{g})_0$ (le bloc associé à λ) ; elle contient donc des objets projectifs.

L'idée principale de la preuve est la suivante : en utilisant des foncteurs de translation, il suffit d'établir ce résultat pour les objets simples associés aux poids dans l'alcôve fondamentale ; et on peut traiter explicitement ces objets en généralisant les calculs du chapitre I pour SL(2) et SL(3). Dans l'étape de réduction à l'alcôve fondamentale, nous utilisons la conjecture de Lusztig sur les caractères des G-modules simples ([Lus80b]). Cette conjecture a été démontrée, grâce à des travaux de Kazhdan-Lusztig ([KL93a], [KL93b], [KL94a], [KL94b], [Lus94]), Kashiwara-Tanisaki ([KT95], [KT96]) et Andersen-Jantzen-Soergel ([AJS94]), lorsque la caractéristique p est suffisamment grande, sans borne explicite. Ceci explique notre restriction sur p.

De ce résultat découle en particulier l'existence d'une graduation de Koszul sur l'algèbre $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$, pour tout caractère λ régulier, sous les mêmes hypothèses que dans [AJS94], c'est-à-dire pour p suffisamment grand (voir la sous-partie 1.4). Nos méthodes sont très différentes de celles utilisées dans l'article [AJS94], et fournissent également des informations sur l'anneau dual, qu'on peut relier à la catégorie $\mathrm{Mod}_0((\mathcal{U}\mathfrak{g})^\lambda)$.

En utilisant un "analogue parabolique" des constructions précédentes, nous démontrons également que pour un poids μ singulier, le bloc $(\mathcal{U}\mathfrak{g})_0^{\hat{\mu}}$ de $(\mathcal{U}\mathfrak{g})_0$ associé à μ peut être muni d'une graduation de Koszul si $p \gg 0$. Dans ce cas aussi nous donnons des informations sur l'anneau dual.

Il découle en particulier que, pour p suffisamment grand, l'anneau $(\mathcal{U}\mathfrak{g})_0$ peut être muni d'une graduation de Koszul. Notons que cette propriété a été conjecturée par Soergel (sous l'hypothèse $p > h$) dans son exposé à l'I.C.M. de Zurich (voir [Soe94]).

2.4 Dualité de Koszul linéaire

Dans le chapitre IV nous présentons une version légèrement différente, et dans un cadre plus général, d'un résultat intermédiaire du chapitre III, obtenue en collaboration avec Ivan Mirković. Ce chapitre est indépendant des trois autres, et ne fait intervenir ni groupe algébrique, ni algèbre de Lie.

La notion de *dg-schéma* a été introduite par Ciocan-Fontanine et Kapranov dans [CFK01]. Pour nous, un dg-schéma sera la donnée d'une paire $(X, \mathcal{A}_X)$, où X est un schéma noethérien et $\mathcal{A}_X$ est un faisceau de $\mathcal{O}_X$-dg-algèbres commutatives (au sens gradué), quasi-cohérent comme $\mathcal{O}_X$-module, et concentré en degrés négatifs ou nuls. Un des grands intérêts de cette notion est le fait que la catégorie dérivée des faisceaux de $\mathcal{A}_X$-dg-modules ne dépend (à équivalence près) du choix de $\mathcal{A}_X$ qu'à quasi-isomorphisme près. Ceci permet de "définir" l'*intersection dérivée* $Y \overset{R}{\cap}_X Z$ de deux sous-schémas fermés Y et Z d'un schéma X comme étant le dg-schéma

$$(X,\, \mathcal{O}_Y \overset{L}{\otimes}_{\mathcal{O}_X} \mathcal{O}_Z),$$

à quasi-isomorphisme près.

Considérons un schéma X noethérien, intègre, séparé et régulier. Soient E un fibré vectoriel au-dessus de X, et $F_1, F_2 \subset E$ des sous-fibrés. Notons $F_1^{\perp}, F_2^{\perp}$ les orthogonaux de F_1, F_2, qui sont des sous-fibrés du dual E^* de E. Avec ces notations, nous établissons une équivalence de catégories contravariante entre une "version graduée" de la catégorie dérivée des dg-faisceaux cohérents sur $F_1 \overset{R}{\cap}_E F_2$, et une "version graduée" de la catégorie dérivée des dg-faisceaux cohérents sur $F_1^{\perp} \overset{R}{\cap}_{E^*} F_2^{\perp}$.

Cette *dualité de Koszul linéaire* généralise la dualité classique entre les modules sur l'algèbre symétrique $\mathrm{S}(V)$ d'un espace vectoriel V (de dimension finie) et l'algèbre extérieure $\Lambda(V^*)$ du dual V^*. Plus précisément, nous remplaçons V par un complexe de faisceaux localement libres de rang fini, ayant deux termes non-nuls.

Plusieurs applications de cette construction en théorie des représentations seront présentées dans un travail ultérieur (voir la sous-partie 3.2 ci-dessous).

Notons que la construction des catégories $\mathcal{C}^{\mathrm{gr}}$, $\mathcal{D}^{\mathrm{gr}}$ considérées en 2.3 est basée sur les mêmes idées que celles développées dans ce chapitre. Cependant, la construction précise de l'équivalence κ dans le chapitre III est légèrement différente des constructions du chapitre IV (et donc n'en est pas un cas particulier).

3 Perspectives

3.1 Action du groupe de tresses précisée

Conjecturalement, l'action du groupe B'_{aff} sur la catégorie $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$ (voir 2.2) admet une description plus précise si p est très bon pour G.

Il existe une application canonique $W \hookrightarrow B'_{\mathrm{aff}}$, qui envoie un élément $w \in W$ sur un élément de B'_{aff} que nous noterons T_w. Pour tout $w \in W$, notons Z_w l'adhérence dans $\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$ du graphe de l'action de w sur $\widetilde{\mathfrak{g}}_{\mathrm{rs}}$ (voir 2.2). Alors Bezrukavnikov conjecture dans [Bez06b] que l'élément T_w agit par convolution avec le noyau $\mathcal{O}_{Z_w} \in \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}})$. Cet énoncé semble beaucoup plus difficile à démontrer que la simple existence de l'action, dans la mesure où les variétés Z_w n'ont *a priori* aucune propriété de régularité. Par exemple, un point essentiel (et non trivial) de la preuve de l'article [Ric08a] consiste à démontrer que la variété Z_{v_0} est normale (et Cohen-Macaulay) lorsque v_0 est l'élément de plus grande longueur dans le groupe de Weyl d'un sous-groupe parabolique de G de rang 2, de type $\mathbf{A}_2$ ou $\mathbf{B}_2$. Dans le cas de $\mathbf{B}_2$, la variété Z_{v_0} n'est pas Gorenstein.

Supposons que le corps de base est $\mathbb{C}$. Il existe un morphisme naturel $\widetilde{\mathfrak{g}} \to \mathfrak{g}^*$, et la variété $\widetilde{\mathfrak{g}} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}$ a même K-théorie que Z. La description conjecturale précédente de l'action donnerait en particulier, pour chaque $w \in W$, un faisceau cohérent sur $\widetilde{\mathfrak{g}} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}$ dont la classe en K-théorie correspond, via l'isomorphisme $K^{G\times\mathbb{C}^\times}(\widetilde{\mathfrak{g}} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}) \cong K^{G\times\mathbb{C}^\times}(Z) \cong \mathcal{H}'_{\mathrm{aff}}$ (voir 2.2), à l'image de T_w dans $\mathcal{H}'_{\mathrm{aff}}$.

Revenons au cas général. Notons B_0 le sous-groupe de B'_{aff} engendré par les T_w, $w \in W$. Ce groupe est isomorphe au groupe de tresses associé à W. De la description plus précise de l'action découlerait également, en utilisant un théorème de Deligne ([Del97]), qu'on peut définir une action *au sens fort* de B_0 sur $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$, c'est-à-dire que pour tout $b \in B_0$ on peut choisir une auto-équivalence F_b de $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$ dans la classe d'isomorphisme associée à b ci-dessus, et pour tous $b, b' \in B_0$ on peut choisir un isomorphisme de foncteurs $F_b \circ F_{b'} \cong F_{bb'}$, de telle sorte que ces données vérifient certaines relations d'associativité. Il serait alors intéressant d'étudier si cette propriété est vraie pour le groupe B'_{aff} tout entier.

3.2 Applications de la dualité de Koszul linéaire

Dans l'introduction du chapitre IV nous présentons deux applications de la dualité de Koszul linéaire, qui seront démontrées dans un travail ultérieur.

Tout d'abord, considérons un groupe algébrique semi-simple, connexe et simplement connexe G sur $\mathbb{C}$, et utilisons les mêmes notations que ci-dessus. On a les sous-fibrés vectoriels $F_1 := \widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}$ et $F_2 := \Delta\mathfrak{g}^* \times (\mathcal{B} \times \mathcal{B})$ du fibré vectoriel constant E au-dessus de $\mathcal{B} \times \mathcal{B}$, de fibre $\mathfrak{g}^* \times \mathfrak{g}^*$. Ici $\Delta\mathfrak{g}^* \subset \mathfrak{g}^* \times \mathfrak{g}^*$ est la copie diagonale. Via la forme de Killing, $\mathfrak{g}$ s'identifie naturellement à $\mathfrak{g}^*$, ce qui identifie également E et E^*. Via cette identification, l'orthogonal $F_1^\perp$ s'identifie à $\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$. L'orthogonal $F_2^\perp$ s'identifie, lui, à la copie anti-diagonale de $\mathfrak{g}^*$. Quitte à multiplier par -1 dans la deuxième copie de $\mathfrak{g}^*$, on peut supposer que $F_2^\perp = \Delta\mathfrak{g}^* \times (\mathcal{B} \times \mathcal{B})$.

Une version équivariante de la construction du chapitre IV donne donc une équivalence entre des catégories de dg-faisceaux cohérent $G \times \mathbb{C}^\times$-équivariants sur

$$(\widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}) \overset{R}{\cap}_{(\mathfrak{g}^* \times \mathfrak{g}^*) \times (\mathcal{B} \times \mathcal{B})} \Delta \mathfrak{g}^* \times (\mathcal{B} \times \mathcal{B}) \quad \text{et} \quad (\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}) \overset{R}{\cap}_{(\mathfrak{g}^* \times \mathfrak{g}^*) \times (\mathcal{B} \times \mathcal{B})} \Delta \mathfrak{g}^* \times (\mathcal{B} \times \mathcal{B}).$$

Notons que, à gauche, l'intersection non-dérivée est Z, et que, à droite, l'intersection non-dérivée est $\widetilde{\mathfrak{g}} \times_{\mathfrak{g}^*} \widetilde{\mathfrak{g}}$. On en déduit aisément que les deux catégories considérées ont des groupes de Grothendieck isomorphes à $K^{G \times \mathbb{C}^\times}(Z) \cong \mathcal{H}'_{\mathrm{aff}}$.

Cette équivalence sera l'ingrédient essentiel d'une réalisation géométrique de l'involution de Iwahori-Matsumoto, c'est-à-dire la construction géométrique d'une équivalence entre les deux catégories ci-dessus telle que l'automorphisme induit en K-théorie est l'involution de Iwahori-Matsumoto. Notons qu'une réalisation géométrique avait été construite par Evens et Mirković pour l'involution de Iwahori-Matsumoto de l'algèbre de Hecke affine étendue *graduée*, dans [EM97].

Pour la deuxième application, considérons un fibré quelconque E, et deux sous-fibrés F_1, F_2, sur le corps de base $\mathbb{C}$. De même que ci-dessus, le groupe de Grothendieck des catégories reliées par la dualité de Koszul linéaire est respectivement $K^{\mathbb{C}^\times}(F_1 \cap F_2)$ et $K^{\mathbb{C}^\times}(F_1^\perp \cap F_2^\perp)$. Dans les deux cas, $\mathbb{C}^\times$ agit par dilatation dans les fibres de E ou E^*. On obtient donc un isomorphisme

$$K^{\mathbb{C}^\times}(F_1 \cap F_2) \ \cong \ K^{\mathbb{C}^\times}(F_1^\perp \cap F_2^\perp).$$

Cet isomorphisme est relié, via le caractère de Chern, à l'isomorphisme en homologie de Borel-Moore

$$\mathcal{H}_*^{\mathrm{BM}}(F_1 \cap F_2) \cong \mathcal{H}_*^{\mathrm{BM}}(F_1^\perp \cap F_2^\perp)$$

défini par Kashiwara en utilisant une transformée de Fourier.

3.3 Koszulité de certaines algèbres associées aux slices de Slodowy

Bezrukavnikov définit dans [Bez06b] une algèbre A_χ, associée à un élément nilpotent $\chi \in \mathfrak{g}^*$. Plus précisément, cette algèbre est associée au slice de Slodowy associé à χ. Il demande dans [Bez06b, 2.26] si cette algèbre peut être munie d'une graduation de Koszul.

Dans le cas où χ est régulier, il est facile de voir que la réponse est positive. Le cas où χ est sous-régulier peut également être traité par des méthodes spécifiques, et la réponse est encore positive. Le cas où $\chi = 0$ est essentiellement traité dans le chapitre III. Mais le problème reste ouvert dans les autres cas. Il pourrait peut-être être traité en comparant la t-structure sur $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$ provenant de l'équivalence $\epsilon_{\mathcal{B}}^0$ de 2.1 et la t-structure *exotique* sur $\mathcal{D}^b\mathrm{Coh}^G(\widetilde{\mathcal{N}})$ définie par Bezrukavnikov dans [Bez06a], puis en se ramenant à une question dans cette dernière catégorie.

3.4 Généralisation aux algèbres de Kac-Moody

Pour le corps de base $\mathbb{C}$, la stratégie d'étude des $\mathfrak{g}$-modules (voir le diagramme en 1.2) a été (partiellement) généralisée aux algèbres de Kac-Moody complexes par Kashiwara et

Tanisaki (voir [Kas90], [KT90], [KT95], [KT96] ; voir également [KT98] pour une vue d'ensemble de ces travaux). Suivant une suggestion de Vasserot, il serait intéressant d'étudier la possibilité d'une généralisation de l'étude de [BMR08], [BMR06] au cas des algèbres de Kac-Moody (ou au moins des algèbres affines) en caractéristique positive. Il semble raisonnable d'espérer que la partie "algébrique" de la construction (c'est-à-dire la relation entre $\mathfrak{g}$-modules et $\mathcal{D}$-modules sur la variété des drapeaux) se généralise sans grand changement, en adaptant les idées de Kashiwara et Tanisaki. Toutefois, trouver un équivalent de la partie "géométrique" de la preuve (c'est-à-dire la relation entre $\mathcal{D}$-modules sur la variété des drapeaux et faisceaux cohérents sur $\widetilde{\mathfrak{g}}^{(1)}$) semble moins clair.

Il serait certainement nécessaire dans cette optique de développer l'étude des représentations des algèbres de Kac-Moody en caractéristique positive, ce qui a été peu fait jusqu'à présent (voir cependant les articles de Mathieu [Mat96] et [Mat03]).

En 2.2 nous avons expliqué que l'action du groupe B'_{aff} sur $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$ est une "catégorification" de l'isomorphisme $\mathcal{H}'_{\mathrm{aff}} \cong K^{G\times\mathbb{C}^\times}(Z)$. Vasserot a donné dans [Vas05] une généralisation partielle de cet isomorphisme au cadre affine. Dans ce cadre, $\mathcal{H}'_{\mathrm{aff}}$ est remplacée par l'algèbre de Cherednik (ou algèbre de Hecke doublement affine), et Z par une *variété de Steinberg affine* Z_{aff} (qui est de dimension infinie). Plus précisément, il donne une généralisation de la construction géométrique des $\mathcal{H}'_{\mathrm{aff}}$-modules simples, qui est elle-même une conséquence de l'isomorphisme $\mathcal{H}'_{\mathrm{aff}} \cong K^{G\times\mathbb{C}^\times}(Z)$. Il serait également intéressant d'étudier une possible "catégorification" de cette construction. Notons cependant que la "bonne" définition de la K-théorie de la variété Z_{aff}, ou de la catégorie $\mathcal{D}^b\mathrm{Coh}(Z_{\mathrm{aff}})$, n'est pas claire dans ce cadre.

3.5 Liens avec les travaux de Premet

Pour finir, il pourrait être fructueux de comparer les constructions des articles [BMR08], [BMR06] aux travaux de Premet sur les représentations des algèbres de Lie en caractéristique positive (voir par exemple [Pre02]). Son approche est basée sur l'étude d'algèbres qui sont des "quantifications" des slices de Slodowy.

Remarque sur les références

Nous utiliserons la convention standard pour faire référence à un énoncé. C'est-à-dire, à l'intérieur, disons, du chapitre III, la référence "Theorem 9.2.1" renvoie au théorème 9.2.1 de ce chapitre III, tandis que la référence "Theorem I.1.2.1" renvoie au théorème 1.2.1 du chapitre I.

Chapter I

Localization in positive characteristic

In this chapter we review the localization theory in positive characteristic due to Bezrukavnikov, Mirković and Rumynin (section 1). Then we perform some explicit computations in the cases $G = \mathrm{SL}(2, \Bbbk)$ (section 2) and $G = \mathrm{SL}(3, \Bbbk)$ (section 3).

Section 3 is a joint work with Roman Bezrukavnikov. It was published as an appendix to [BMR08][1].

1 Review of the results of [BMR08] and [BMR06]

1.1 Notation

Let $\Bbbk$ be an algebraically closed field of characteristic p. Most of the time (and in particular in this chapter), p is assumed to be positive. However, sometimes in chapters II and IV it can be 0.

Let R be a root system, and G be the corresponding connected, semi-simple, simply-connected algebraic group over $\Bbbk$. We denote by h the Coxeter number of G. Let B be a Borel subgroup of G, $T \subset B$ a maximal torus, U the unipotent radical of B, B^+ the Borel subgroup opposite to B, and U^+ its unipotent radical. Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, $\mathfrak{n}$, $\mathfrak{b}^+$, $\mathfrak{n}^+$ be their respective Lie algebras. Let $R^+ \subset R$ be the positive roots, chosen as the roots in $\mathfrak{n}^+$, and Φ be the corresponding set of simple roots. As usual, we denote by ρ the half sum of the positive roots.

We denote by $U_\alpha \subset G$ the image of the one-parameter subgroup attached to the root α. Let $\mathcal{B} := G/B$ be the flag variety of G, and $\widetilde{\mathcal{N}} := T^*\mathcal{B}$ be its cotangent bundle. We have the geometric description

$$\widetilde{\mathcal{N}} = \{(X, gB) \in \mathfrak{g}^* \times \mathcal{B} \mid X_{|g \cdot \mathfrak{b}} = 0\}.$$

We will also consider the "extended cotangent bundle"

$$\widetilde{\mathfrak{g}} := \{(X, gB) \in \mathfrak{g}^* \times \mathcal{B} \mid X_{|g \cdot \mathfrak{n}} = 0\}.$$

[1] Note however that the normalization of the splitting bundles here is different from the one chosen in the appendix to [BMR08].

Let $\mathfrak{h}$ denote the "abstract" Cartan subalgebra of $\mathfrak{g}$, isomorphic to $\mathfrak{b}_0/[\mathfrak{b}_0, \mathfrak{b}_0]$ for any Borel subalgebra $\mathfrak{b}_0$ of $\mathfrak{g}$. Then there is a natural morphism $\widetilde{\mathfrak{g}} \to \mathfrak{h}^*$, which sends a pair (X, gB) to $X_{|g\cdot\mathfrak{b}}$, an element of the dual of $g \cdot \mathfrak{b}/g \cdot \mathfrak{n} \cong \mathfrak{h}$. The Lie algebras $\mathfrak{t}$ and $\mathfrak{h}$ are naturally isomorphic, via the morphism $\mathfrak{t} \xrightarrow{\sim} \mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$.

For each positive root α, we choose isomorphisms of algebraic groups $u_\alpha : \Bbbk \xrightarrow{\sim} U_\alpha$ and $u_{-\alpha} : \Bbbk \xrightarrow{\sim} U_{-\alpha}$ such that for all $t \in T$ we have $t \cdot u_\alpha(x) \cdot t^{-1} = u_\alpha(\alpha(t)x)$ and $t \cdot u_{-\alpha}(x) \cdot t^{-1} = u_{-\alpha}(\alpha(t)^{-1}x)$, and such that these morphisms extend to a morphism of algebraic groups $\psi_\alpha : \mathrm{SL}(2, \Bbbk) \to G$ such that

$$\psi_\alpha \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = u_\alpha(x), \quad \psi_\alpha \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = u_{-\alpha}(x).$$

We define the elements

$$e_\alpha := d(u_\alpha)_0(1), \quad e_{-\alpha} := d(u_{-\alpha})_0(1), \quad h_\alpha := [e_\alpha, e_{-\alpha}].$$

We denote by $\mathbb{Y} := \mathbb{Z}R$ the root latice of R, and by $\mathbb{X} := X^*(T)$ the weight lattice. Let W be the Weyl group of (G, T), $W_{\mathrm{aff}} := W \ltimes \mathbb{Y}$ be the affine Weyl group, and $W'_{\mathrm{aff}} := W \ltimes \mathbb{X}$ be the extended affine Weyl group. They act naturally on $\mathbb{X}$. We denote by "$\bullet$" the dot-action of W'_{aff} on $\mathbb{X}$, defined by $w \bullet \lambda = w(\lambda + \rho) - \rho$.

For $\lambda \in \mathbb{X}$ a dominant weight, we denote by $L(\lambda)$ the simple G-module with highest weight λ, and by $\mathrm{Ind}_B^G(\lambda)$ the corresponding induced module. For a general $\lambda \in \mathbb{X}$, we denote by $\mathcal{O}_\mathcal{B}(\lambda)$ the line bundle on $\mathcal{B}$ naturally associated to λ (see *e.g.* [Jan03, I.5.8]).

If $P \subseteq G$ is a parabolic subgroup containing B, $\mathfrak{p}$ its Lie algebra, $\mathfrak{p}^{\mathrm{u}}$ the nilpotent radical of $\mathfrak{p}$, and $\mathcal{P} = G/P$ the corresponding flag variety, we consider the following analogue of the variety $\widetilde{\mathfrak{g}}$:

$$\widetilde{\mathfrak{g}}_\mathcal{P} := \{(X, gP) \in \mathfrak{g}^* \times \mathcal{P} \mid X_{|g\cdot\mathfrak{p}^{\mathrm{u}}} = 0\}.$$

In particular, $\widetilde{\mathfrak{g}}_\mathcal{B} = \widetilde{\mathfrak{g}}$. The quotient morphism $\pi_\mathcal{P} : \mathcal{B} \to \mathcal{P}$ induces a morphism

$$\widetilde{\pi}_\mathcal{P} : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}_\mathcal{P}. \tag{1.1.1}$$

In this situation, we also denote by $W_P \subseteq W$ the Weyl group of P.

If $\alpha \in \Phi$, and P_α is the minimal parabolic subgroup containing B associated to α, we simplify the notation by setting $\widetilde{\mathfrak{g}}_\alpha := \widetilde{\mathfrak{g}}_{G/P_\alpha}$, $\widetilde{\pi}_\alpha := \widetilde{\pi}_{G/P_\alpha}$.

For $\chi \in \mathfrak{g}^*$ nilpotent we define $\mathcal{B}_\chi$, respectively $\mathcal{P}_\chi$, as the set-theoretical inverse image of χ under $\widetilde{\mathfrak{g}} \to \mathfrak{g}^*$, respectively $\widetilde{\mathfrak{g}}_\mathcal{P} \to \mathfrak{g}^*$, endowed with the reduced scheme structure. The variety $\mathcal{B}_\chi$ is isomorphic to the *Springer fiber* associated to χ.

If X is a scheme, and $Y \subset X$ a closed subscheme, one says that an $\mathcal{O}_X$-module $\mathcal{F}$ is *supported on* Y if $\mathcal{F}_x = 0$ for $x \notin Y$. If $\mathcal{F}$ is coherent, this is equivalent to requiring that the ideal sheaf of Y in $\mathcal{O}_X$ acts nilpotently on $\mathcal{F}$. We write $\mathrm{Coh}_Y(X)$ for the full subcategory of the category $\mathrm{Coh}(X)$ of coherent sheaves on X whose objects are supported on Y.

1.2 Localization theorem

From now on in this chapter we assume that $p > h$.

Let $\mathfrak{Z}$ be the center of $\mathcal{U}\mathfrak{g}$, the enveloping algebra of $\mathfrak{g}$. The subalgebra of G-invariants, $\mathfrak{Z}_{\mathrm{HC}} := (\mathcal{U}\mathfrak{g})^G$ is central in $\mathcal{U}\mathfrak{g}$. This is the "Harish-Chandra part" of the center, which is isomorphic to $S(\mathfrak{t})^{(W,\bullet)}$, the algebra of W-invariants in the symmetric algebra of $\mathfrak{t}$, for the dot-action. The center $\mathfrak{Z}$ also has an other part, the "Frobenius part" $\mathfrak{Z}_{\mathrm{Fr}}$, which is generated, as an algebra, by the elements $X^p - X^{[p]}$ for $X \in \mathfrak{g}$. It is isomorphic to $S(\mathfrak{g}^{(1)})$, the functions on the Frobenius twist of $\mathfrak{g}^*$. Under our assumption $p > h$, there is an isomorphism (see *e.g.* [MR99])

$$\mathfrak{Z}_{\mathrm{HC}} \otimes_{\mathfrak{Z}_{\mathrm{Fr}} \cap \mathfrak{Z}_{\mathrm{HC}}} \mathfrak{Z}_{\mathrm{Fr}} \xrightarrow{\sim} \mathfrak{Z}.$$

Hence, a character of $\mathfrak{Z}$ is given by a "compatible pair" $(\nu, \chi) \in \mathfrak{t}^* \times \mathfrak{g}^{*(1)}$. For simplicity, here we will only consider the case when χ is nilpotent, and $\nu \in \mathfrak{t}^*$ is *integral*, *i.e.* in the image of the natural map $\mathbb{X} \to \mathfrak{t}^*$ (such a pair is always "compatible"). If $\lambda \in \mathbb{X}$, we still denote by λ its image in $\mathfrak{t}^*$. We denote the corresponding specializations by

$$\begin{aligned}(\mathcal{U}\mathfrak{g})^\lambda &:= (\mathcal{U}\mathfrak{g}) \otimes_{\mathfrak{Z}_{\mathrm{HC}}} \Bbbk_\lambda,\\ (\mathcal{U}\mathfrak{g})_\chi &:= (\mathcal{U}\mathfrak{g}) \otimes_{\mathfrak{Z}_{\mathrm{Fr}}} \Bbbk_\chi,\\ (\mathcal{U}\mathfrak{g})^\lambda_\chi &:= (\mathcal{U}\mathfrak{g}) \otimes_{\mathfrak{Z}} \Bbbk_{(\lambda,\chi)}.\end{aligned}$$

Let $\mathrm{Mod}^{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$ be the abelian category of finitely generated $\mathcal{U}\mathfrak{g}$-modules. If $\lambda \in \mathbb{X}$ and $\chi \in \mathfrak{g}^{*(1)}$ is nilpotent, we denote by $\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,\chi)}(\mathcal{U}\mathfrak{g})$ the abelian category of finitely generated $\mathcal{U}\mathfrak{g}$-modules on which $\mathfrak{Z}$ acts with generalized character (λ, χ). We define similarly the categories $\mathrm{Mod}^{\mathrm{fg}}_\chi((\mathcal{U}\mathfrak{g})^\lambda)$, $\mathrm{Mod}^{\mathrm{fg}}_\lambda((\mathcal{U}\mathfrak{g})_\chi)$, $\mathrm{Mod}^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^\lambda_\chi)$. We also denote by $\mathrm{Mod}^{\mathrm{fg}}_\lambda(\mathcal{U}\mathfrak{g})$ the category of finitely generated $\mathcal{U}\mathfrak{g}$-modules on which $\mathfrak{Z}_{\mathrm{HC}}$ acts with generalized character λ. Hence we have inclusions

$$\begin{tikzcd}
 & \mathrm{Mod}^{\mathrm{fg}}_\chi((\mathcal{U}\mathfrak{g})^\lambda) \arrow[dr, hook] & & \mathrm{Mod}^{\mathrm{fg}}_\lambda(\mathcal{U}\mathfrak{g}) \arrow[d, hook] \\
\mathrm{Mod}^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^\lambda_\chi) \arrow[ur, hook] \arrow[dr, hook] & & \mathrm{Mod}^{\mathrm{fg}}_{(\lambda,\chi)}(\mathcal{U}\mathfrak{g}) \arrow[r, hook] \arrow[ur, hook] & \mathrm{Mod}^{\mathrm{fg}}(\mathcal{U}\mathfrak{g}) \\
 & \mathrm{Mod}^{\mathrm{fg}}_\lambda((\mathcal{U}\mathfrak{g})_\chi) \arrow[ur, hook] & &
\end{tikzcd}$$

Recall that a weight $\lambda \in \mathbb{X}$ is called *regular* if, for any root α, $\langle \lambda + \rho, \alpha^\vee \rangle \notin p\mathbb{Z}$, *i.e.* if λ is not on any reflection hyperplane of W_{aff} (for the dot-action). If $\mu \in \mathbb{X}$, we denote by $\mathrm{Stab}_{(W_{\mathrm{aff}},\bullet)}(\mu)$ the stabilizer of μ for the dot-action of W_{aff} on $\mathbb{X}$. Under our hypothesis $p > h$, we have $(p\mathbb{X}) \cap \mathbb{Y} = p\mathbb{Y}$. It follows that $\mathrm{Stab}_{(W_{\mathrm{aff}},\bullet)}(\mu)$ is also the stabilizer of μ for the action of W'_{aff} on $\mathbb{X}$.

By the work of Bezrukavnikov, Mirković and Rumynin, we have (see [BMR08, 5.3.1] for (i), and [BMR06, 1.5.1.c, 1.5.2.b] for (ii)):

Theorem 1.2.1. (i) *Let $\lambda \in \mathbb{X}$ be regular, and $\chi \in \mathfrak{g}^*$ be nilpotent. There exist equivalences of categories*

$$\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}_\chi^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}) \cong \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,\chi)}(\mathcal{U}\mathfrak{g}), \tag{1.2.2}$$

$$\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}_\chi^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \cong \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{\chi}((\mathcal{U}\mathfrak{g})^\lambda). \tag{1.2.3}$$

(ii) *More generally, let $\mu \in \mathbb{X}$, and let P be a parabolic subgroup of G containing B such that*[2] $\mathrm{Stab}_{(W_{\mathrm{aff}},\bullet)}(\mu) = W_P$. *Let $\mathcal{P} = G/P$ be the corresponding flag variety. Then there exists an equivalence of categories*

$$\mathcal{D}^b\mathrm{Coh}_{\mathcal{P}_\chi^{(1)}}(\widetilde{\mathfrak{g}}_{\mathcal{P}}^{(1)}) \cong \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\mu,\chi)}(\mathcal{U}\mathfrak{g}).$$

Let us recall briefly how equivalence (1.2.2) can be constructed. Here we use the notation of [BMR08]. Consider the sheaf of algebras $\widetilde{\mathcal{D}}$ on $\mathcal{B}$; it can also be considered as a sheaf of algebras on $\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$, and it is an Azumaya algebra on this space (see [BMR08, 3.1.3]). Here the morphism $\mathfrak{h}^* \to \mathfrak{h}^{*(1)}$ is the *Artin-Schreier map* (see [BMR08]).

We denote by $\mathrm{Mod}^{\mathrm{c}}(\widetilde{\mathcal{D}})$ the category of quasi-coherent, locally finitely generated $\widetilde{\mathcal{D}}$-modules (either on $\mathcal{B}$, or on $\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$; this is equivalent). For $\nu \in \mathfrak{t}^* \cong \mathfrak{h}^*$ we denote by $\mathrm{Mod}^{\mathrm{c}}_\nu(\widetilde{\mathcal{D}})$, resp. $\mathrm{Mod}^{\mathrm{c}}_{(\nu,\chi)}(\widetilde{\mathcal{D}})$, the full subcategory of $\mathrm{Mod}^{\mathrm{c}}(\widetilde{\mathcal{D}})$ whose objects are supported on $\widetilde{\mathcal{N}}^{(1)} \times \{\nu\} \subset \widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$, respectively on $\mathcal{B}_\chi^{(1)} \times \{\nu\} \subset \widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$. If $\lambda \in \mathbb{X}$ is regular, the functor $R\Gamma : \mathcal{D}^b\mathrm{Mod}^{\mathrm{c}}_\lambda(\widetilde{\mathcal{D}}) \to \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_\lambda(\mathcal{U}\mathfrak{g})$ is an equivalence of categories. Its inverse is the localization functor $\mathcal{L}^{\hat{\lambda}}$. These functors restrict to equivalences between $\mathcal{D}^b\mathrm{Mod}^c_{(\lambda,\chi)}(\widetilde{\mathcal{D}})$ and $\mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,\chi)}(\mathcal{U}\mathfrak{g})$ for any nilpotent $\chi \in \mathfrak{g}^{*(1)}$.

Next, the Azumaya algebra $\widetilde{\mathcal{D}}$ splits on the formal neighborhood of $\mathcal{B}_\chi^{(1)} \times \{\lambda\}$ in $\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$. Hence, the choice of a splitting bundle on this formal neighborhood yields an equivalence of categories $\mathrm{Coh}_{\mathcal{B}_\chi^{(1)} \times \{\lambda\}}(\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*) \cong \mathrm{Mod}^c_{(\lambda,\chi)}(\widetilde{\mathcal{D}})$. Finally, as remarked in [BMR06, 1.5.3.c], the projection $\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^* \to \widetilde{\mathfrak{g}}^{(1)}$ induces an isomorphism between the formal neighborhood of $\mathcal{B}_\chi^{(1)} \times \{\lambda\}$ and the formal neighborhood of $\mathcal{B}_\chi^{(1)}$. This isomorphism induces an equivalence of categories $\mathrm{Coh}_{\mathcal{B}_\chi^{(1)} \times \{\lambda\}}(\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*) \cong \mathrm{Coh}_{\mathcal{B}_\chi^{(1)}}(\widetilde{\mathfrak{g}}^{(1)})$.

These equivalences yield the desired equivalence (1.2.2).

We choose the normalizations of the splitting bundles as in [BMR06, 1.3.5], and denote by

$$\gamma^{\mathcal{B}}_{(\lambda,\chi)} : \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}_\chi^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}) \xrightarrow{\sim} \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,\chi)}(\mathcal{U}\mathfrak{g})$$

the equivalence associated to a regular $\lambda \in \mathbb{X}$ and a nilpotent $\chi \in \mathfrak{g}^{*(1)}$. We also denote by $\mathcal{M}^{\mathcal{B}}_{(\lambda,\chi)}$ the splitting bundle associated to (λ,χ). Similarly, for $\lambda, \mu, \mathcal{P}$ as in Theorem

[2]Equivalently, this means that μ is on the reflection hyperplane corresponding to any simple root of W_P, but not on any hyperplane of a reflection (simple or not) in $W_{\mathrm{aff}} - W_P$.

1.2.1, we denote by

$$\epsilon^{\mathcal{B}}_{(\lambda,\chi)} : \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}_\chi}(\widetilde{\mathcal{N}}^{(1)}) \xrightarrow{\sim} \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_\chi((\mathcal{U}\mathfrak{g})^\lambda),$$
$$\gamma^{\mathcal{P}}_{(\mu,\chi)} : \mathcal{D}^b\mathrm{Coh}_{\mathcal{P}^{(1)}_\chi}(\widetilde{\mathfrak{g}}^{(1)}_{\mathcal{P}}) \xrightarrow{\sim} \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\mu,\chi)}(\mathcal{U}\mathfrak{g})$$

the equivalences obtained with the normalizations of [BMR06, 1.3.5].

If $\chi = 0$, we simplify the notation by writing $\gamma^{\mathcal{B}}_\lambda$, $\epsilon^{\mathcal{B}}_\lambda$, $\gamma^{\mathcal{P}}_\lambda$ instead of $\gamma^{\mathcal{B}}_{(\lambda,0)}$, $\epsilon^{\mathcal{B}}_{(\lambda,0)}$, $\gamma^{\mathcal{P}}_{(\lambda,0)}$. In this case $\mathcal{B}_0$ is just the zero-section of $\widetilde{\mathfrak{g}}$, which we write $\mathcal{B}$. We also write $\mathcal{M}^\lambda$ for $\mathcal{M}^{\mathcal{B}}_{(\lambda,0)}$.

If $\lambda \in \mathbb{X}$ is regular and $\nu \in \mathbb{X}$, then $\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,\chi)}(\mathcal{U}\mathfrak{g})$ and $\mathrm{Mod}^{\mathrm{fg}}_{(\lambda+p\nu,\chi)}(\mathcal{U}\mathfrak{g})$ coincide. But the equivalences $\gamma^{\mathcal{B}}_{(\lambda,\chi)}$ and $\gamma^{\mathcal{B}}_{(\lambda+p\nu,\chi)}$ differ by a shift: $\gamma^{\mathcal{B}}_{(\lambda+p\nu,\chi)}(\mathcal{F}) = \gamma^{\mathcal{B}}_{(\lambda,\chi)}(\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}(\nu) \otimes_{\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}} \mathcal{F})$ for $\mathcal{F}$ in $\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}_\chi}(\widetilde{\mathfrak{g}}^{(1)})$.

1.3 Translation functors

Let us fix a nilpotent $\chi \in \mathfrak{g}^{*(1)}$. For $\lambda, \mu \in \mathbb{X}$, the translation functor

$$T^\mu_\lambda : \mathrm{Mod}^{\mathrm{fg}}_{(\lambda,\chi)}(\mathcal{U}\mathfrak{g}) \to \mathrm{Mod}^{\mathrm{fg}}_{(\mu,\chi)}(\mathcal{U}\mathfrak{g})$$

is defined in [BMR08, 6.1]. Let us recall the geometric counterparts of these functors. Let P be a parabolic subgroup of G containing B and let $\mathcal{P} = G/P$. By [BMR06, 2.2.5] we have:

Proposition 1.3.1. *Let $\lambda \in \mathbb{X}$ be regular, and let $\mu \in \mathbb{X}$ be in the closure of the facet of λ. Assume that $\mathrm{Stab}_{(W_{\mathrm{aff}},\bullet)}(\mu) = W_P$ (with the same notation as in Theorem 1.2.1(ii)). There exist isomorphisms of functors*

$$T^\mu_\lambda \circ \gamma^{\mathcal{B}}_{(\lambda,\chi)} \cong \gamma^{\mathcal{P}}_{(\mu,\chi)} \circ R(\widetilde{\pi}_{\mathcal{P}})_* \quad \textit{and} \quad T^\lambda_\mu \circ \gamma^{\mathcal{P}}_{(\mu,\chi)} \cong \gamma^{\mathcal{B}}_{(\lambda,\chi)} \circ L(\widetilde{\pi}_{\mathcal{P}})^*.$$

1.4 Sheaves on the zero-section

In this subsection we restrict to the case $\chi = 0$, $\lambda = 0$ (hence λ is regular). By Theorem 1.2.1 we have equivalences of categories

$$\epsilon^{\mathcal{B}}_0 : \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \xrightarrow{\sim} \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_0((\mathcal{U}\mathfrak{g})^\lambda),$$
$$\gamma^{\mathcal{B}}_0 : \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}) \xrightarrow{\sim} \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(0,0)}(\mathcal{U}\mathfrak{g}).$$

Let $i : \widetilde{\mathcal{N}}^{(1)} \hookrightarrow \widetilde{\mathfrak{g}}^{(1)}$, $j : \mathcal{B}^{(1)} \hookrightarrow \widetilde{\mathcal{N}}^{(1)}$, $k : \mathcal{B}^{(1)} \hookrightarrow \widetilde{\mathfrak{g}}^{(1)}$ be the natural inclusions. Let also $\mathrm{Fr} : \mathcal{B} \to \mathcal{B}^{(1)}$ be the Frobenius morphism. If $\mathcal{G} \in \mathrm{Coh}(\mathcal{B}^{(1)})$, then $\mathrm{Fr}^*\mathcal{G} \in \mathrm{Coh}(\mathcal{B})$ has a natural structure of $\mathcal{D}^0$-module, coming from the action on $\mathcal{O}_{\mathcal{B}}$. This is the action we consider in the following lemma.

Lemma 1.4.1. *For $\mathcal{F} \in \mathrm{Coh}(\mathcal{B}^{(1)})$ we have isomorphisms*

$$\epsilon^{\mathcal{B}}_0(j_*\mathcal{F}) \cong R\Gamma\big(\mathcal{B}, \mathrm{Fr}^*_{\mathcal{B}}(\mathcal{F}(\rho))\big), \qquad \gamma^{\mathcal{B}}_0(k_*\mathcal{F}) \cong R\Gamma\big(\mathcal{B}, \mathrm{Fr}^*_{\mathcal{B}}(\mathcal{F}(\rho))\big).$$

Proof. We only prove the second isomorphism (the first one can be proved similarly). It is well-known that $(\mathcal{U}\mathfrak{g})_0^{-\rho} \cong \mathrm{End}_{\Bbbk}\big(L((p-1)\rho)\big)$. It follows, by the choice of the splitting bundles (see [BMR06, 1.3.5]), that

$$k^*\mathcal{M}^0 \cong \mathrm{Fr}_*(\mathcal{O}_{\mathcal{B}}(\rho)) \otimes_{\mathrm{Fr}_*\mathcal{O}_{\mathcal{B}}} \big(L((p-1)\rho) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}}\big). \tag{1.4.2}$$

Here the structure of $(\mathcal{U}\mathfrak{g})_0^{-\rho}$-module on $L((p-1)\rho)$ gives an action of $\mathcal{D}^{-\rho}$ on $L((p-1)\rho) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}}$, hence an action of $\mathcal{D}^0$ on $\mathrm{Fr}_*(\mathcal{O}_{\mathcal{B}}(\rho)) \otimes_{\mathrm{Fr}_*\mathcal{O}_{\mathcal{B}}} \big(L((p-1)\rho) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}}\big)$. By Andersen ([And80]) or Haboush ([Hab80]) there is an isomorphism

$$\big(\mathrm{Fr}_*(\mathcal{O}_{\mathcal{B}}(-\rho))\big) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}}(\rho) \cong L((p-1)\rho) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}}. \tag{1.4.3}$$

Here the left-hand side has a natural action of $\mathcal{D}^{-\rho}$, and the isomorphism is compatible with the two $\mathcal{D}^{-\rho}$-module structures. From (1.4.2) and (1.4.3) we deduce an isomorphism

$$(k^*\mathcal{M}^0) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho) \cong \mathrm{Fr}_*\mathcal{O}_{\mathcal{B}}, \tag{1.4.4}$$

where the structure of $\mathcal{D}^0$-module on the right-hand side comes from the natural action on $\mathcal{O}_{\mathcal{B}}$.

Using (1.4.4) and the projection formula, we deduce

$$\begin{aligned}
\gamma_0^{\mathcal{B}}(k_*\mathcal{F}) &\cong R\Gamma\big(\widetilde{\mathfrak{g}}^{(1)}, \mathcal{M}^0 \otimes_{\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}} k_*\mathcal{F}\big) \\
&\cong R\Gamma\big(\mathcal{B}^{(1)}, (k^*\mathcal{M}^0) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{F}\big) \\
&\cong R\Gamma\big(\mathcal{B}^{(1)}, (\mathrm{Fr}_*\mathcal{O}_{\mathcal{B}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} (\mathcal{F}(\rho))\big) \\
&\cong R\Gamma(\mathcal{B}, \mathrm{Fr}^*(\mathcal{F}(\rho))).
\end{aligned}$$

This concludes the proof of the lemma. □

2 The case $G = \mathrm{SL}(2, \Bbbk)$

In this section we perform explicit computations for $G = \mathrm{SL}(2, \Bbbk)$. They will be generalized in III.6.4 and III.7.2 below. Here $p > 2$.

2.1 Notation

We keep the notation of 1.1 and 1.2, with $G = \mathrm{SL}(2, \Bbbk)$. Here $\mathbb{X} \cong \mathbb{Z}$, the unique simple root α corresponds to 2, and ρ corresponds to 1. Moreover there is a natural isomorphism $\mathcal{B} \cong \mathbb{P}^1$ such that $\mathcal{O}_{\mathcal{B}}(n\rho)$ corresponds to $\mathcal{O}_{\mathbb{P}^1}(n)$ for any $n \in \mathbb{Z}$. We denote by $j : \mathcal{B}^{(1)} \hookrightarrow \widetilde{\mathcal{N}}^{(1)}$ the inclusion of the zero-section.

Here we consider the weights $\lambda = 0$, $\chi = 0$. Recall the equivalence of (1.2.3)

$$\epsilon_0^{\mathcal{B}} : \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \xrightarrow{\sim} \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0).$$

By a theorem of Curtis ([Cur60]) and the description of $\mathfrak{Z}$ in 1.2, the simple $\mathcal{U}\mathfrak{g}$-modules in the category $\mathrm{Mod}_{(0,0)}^{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$ are the restrictions of the simple G-modules $L(\lambda)$ for λ a restricted dominant weight in the orbit of 0 under the dot-action of W'_{aff}. These weights are 0 and $p - 2$.

2.2 Simple modules

First, we compute the inverse images under $\epsilon_0^{\mathcal{B}}$ of the simple modules in $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0)$.

Proposition 2.2.1. *The inverse images under $\epsilon_0^{\mathcal{B}}$ of the simple modules in $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0)$ are the following:*

0	$j_*(\mathcal{O}_{(\mathbb{P}^1)^{(1)}}(-1))$
$p-2$	$j_*(\mathcal{O}_{(\mathbb{P}^1)^{(1)}}(-2))[1]$

Proof. By Lemma 1.4.1 we have $\epsilon_0^{\mathcal{B}}(j_*\mathcal{O}_{(\mathbb{P}^1)^{(1)}}(-1)) \cong R\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \cong \Bbbk$, which proves the first line. Similarly we have $\epsilon_0^{\mathcal{B}}(j_*\mathcal{O}_{(\mathbb{P}^1)^{(1)}}(-2)[1]) \cong R\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-p))[1]$. By Serre duality we deduce that $\epsilon_0^{\mathcal{B}}(j_*\mathcal{O}_{(\mathbb{P}^1)^{(1)}}(-2)[1]) \cong \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p-2))^* \cong L(p-2)$. This concludes the proof. $\square$

2.3 Projective covers

The abelian category $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0)$ does not contain any projective object. However, some objects of $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)})$ "behave like" projective modules. If L is a simple object in $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0)$ and $\mathcal{F} \in \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)})$, we say that $\mathcal{F}$ *represents the projective cover of* L if

$$\mathrm{Ext}^n_{\widetilde{\mathcal{N}}^{(1)}}(\mathcal{F}, (\epsilon_0^{\mathcal{B}})^{-1}L) = \begin{cases} \Bbbk & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

and if $\mathrm{Ext}^*_{\widetilde{\mathcal{N}}^{(1)}}(\mathcal{F}, (\epsilon_0^{\mathcal{B}})^{-1}M) = 0$ for any simple object M of $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0)$ not isomorphic to L. Note that if $\mathcal{F}$ is such an object, then the completion of $\mathcal{F}$ to the formal neighborhood of $\mathcal{B}^{(1)}$ in $\widetilde{\mathcal{N}}^{(1)}$ indeed corresponds, under the equivalence of [BMR06, 5.4.1], to a projective module for the completion of $(\mathcal{U}\mathfrak{g})^0$ with respect to the image of the maximal ideal of $\mathfrak{Z}_{\mathrm{Fr}}$ corresponding to the trivial character 0. See also III.6.4 below for other comments on these objects.

In the next proposition we compute objects representing the projective covers of the simple modules for $G = \mathrm{SL}(2, \Bbbk)$ (in particular we show that such objects exist).

Proposition 2.3.1. *The following objects of $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)})$ represent the projective covers of the simple objects of $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0)$:*

0	$\mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}(-1)$
$p-2$	$\mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}$

Proof. By adjunction we have

$$\begin{aligned} \mathrm{Ext}^n_{\widetilde{\mathcal{N}}^{(1)}}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}(-1), j_*(\mathcal{O}_{(\mathbb{P}^1)^{(1)}}(-1))\big) &\cong \mathrm{Ext}^n_{(\mathbb{P}^1)^{(1)}}\big(\mathcal{O}_{(\mathbb{P}^1)^{(1)}}(-1), \mathcal{O}_{(\mathbb{P}^1)^{(1)}}(-1)\big) \\ &\cong \begin{cases} \Bbbk & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly one checks that $\mathrm{Ext}^n_{\widetilde{\mathcal{N}}^{(1)}}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}(-1), j_*(\mathcal{O}_{(\mathbb{P}^1)^{(1)}}(-2))[1]\big) = 0$ for $n \in \mathbb{Z}$, which proves the claim for the weight 0.

Now, similarly, $\mathrm{Ext}^n_{\widetilde{\mathcal{N}}^{(1)}}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}, j_*(\mathcal{O}_{(\mathbb{P}^1)^{(1)}}(-1))\big) = 0$ for $n \in \mathbb{Z}$, and

$$\begin{aligned}\mathrm{Ext}^n_{\widetilde{\mathcal{N}}^{(1)}}\big(\mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}, j_*(\mathcal{O}_{(\mathbb{P}^1)^{(1)}}(-2))[1]\big) &\cong \mathrm{Ext}^{n+1}_{(\mathbb{P}^1)^{(1)}}\big(\mathcal{O}_{(\mathbb{P}^1)^{(1)}}, \mathcal{O}_{(\mathbb{P}^1)^{(1)}}(-2)\big)\\ &\cong \begin{cases} \Bbbk & \text{if } n = 0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

This concludes the proof. □

3 The case $G = \mathrm{SL}(3, \Bbbk)$

In this section, $p > 3$.

3.1 Notation

We keep the notation of 1.1, with $G = \mathrm{SL}(3, \Bbbk)$, and denote by α_1, α_2 the simple roots of G and ω_1, ω_2 the fundamental weights. Let s_i be the reflection $s_{\alpha_i} \in W$. We denote by $\mathcal{B}^{(1)} \xrightarrow{j} \widetilde{\mathcal{N}}^{(1)} \xrightarrow{p} \mathcal{B}^{(1)}$ the inclusion of the zero-section and the natural projection. There are two natural maps[3] $\pi_i : \mathcal{B} \to \mathbb{P}^2$ mapping a flag $0 \subset V_1 \subset V_2 \subset \Bbbk^3$ to $V_j,\ j = 1, 2$. For $n \in \mathbb{Z}$ and $\lambda \in \mathbb{X}$ we have isomorphisms:

$$\pi_i^* \mathcal{O}_{\mathbb{P}^2}(n) \cong \mathcal{O}_{\mathcal{B}}(n\omega_i)\ (i = 1, 2), \qquad \mathrm{Fr}^*\big(\mathcal{O}_{\mathcal{B}^{(1)}}(\lambda)\big) \cong \mathcal{O}_{\mathcal{B}}(p\lambda).$$

Recall also the exact sequence (see [Har77, Theorem II.8.13]):

$$0 \to \Omega^1_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2} \to 0. \tag{3.1.1}$$

As in section 2, we consider the case $\lambda = 0$, $\chi = 0$. We have an equivalence (see (1.2.3))

$$\epsilon_0^{\mathcal{B}} : \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \xrightarrow{\sim} \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0).$$

As is 2.1, the simple $\mathcal{U}\mathfrak{g}$-modules in the category $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0)$ are the restrictions of the simple G-modules $L(\lambda)$ for λ a restricted dominant weight in the orbit of 0 under the dot-action of W'_{aff}. These weights are the following:

$$0,\ (p-3)\omega_1,\ (p-3)\omega_2,\ (p-2)\omega_1 + \omega_2,\ \omega_1 + (p-2)\omega_2,\ (p-2)\rho.$$

3.2 Simple modules

First, we compute the inverse images under $\epsilon_0^{\mathcal{B}}$ of the simple modules in $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0)$.

[3] Here, for simplicity, we choose an identification between the projective space of lines and of planes in $\Bbbk^3$.

Proposition 3.2.1. *The inverse images under $\epsilon_0^{\mathcal{B}}$ of the simple modules in $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0)$ are given by:*

0	$j_*(\mathcal{O}_{\mathcal{B}^{(1)}}(-\rho))$
$(p-3)\omega_2$	$j_*\big(\mathcal{O}_{\mathcal{B}^{(1)}}(-2\omega_1 - \omega_2)\big)[2]$
$(p-3)\omega_1$	$j_*\big(\mathcal{O}_{\mathcal{B}^{(1)}}(-\omega_1 - 2\omega_2)\big)[2]$
$(p-2)\omega_1 + \omega_2$	$j_*\big((\pi_1^{(1)})^*\Omega^1_{(\mathbb{P}^2)^{(1)}}(-\omega_2)\big)[1]$
$\omega_1 + (p-2)\omega_2$	$j_*\big((\pi_2^{(1)})^*\Omega^1_{(\mathbb{P}^2)^{(1)}}(-\omega_1)\big)[1]$
$(p-2)\rho$	$\mathcal{L}$

where $\mathcal{L}$ is the cone of the only (up to a constant) nonzero morphism $j_*\mathcal{O}_{\mathcal{B}^{(1)}}(-\rho) \to j_*\mathcal{O}_{\mathcal{B}^{(1)}}(-2\rho)[3]$.

Proof. By Lemma 1.4.1, $\epsilon_0^{\mathcal{B}}(j_*\mathcal{O}_{\mathcal{B}^{(1)}}(-\rho)) \cong R\Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}) \cong \Bbbk$. This settles the first line.

Similarly, $\epsilon_0^{\mathcal{B}}(j_*\mathcal{O}_{\mathcal{B}^{(1)}}(-2\omega_1 - \omega_2)) \cong R\Gamma(\mathcal{B}, \mathrm{Fr}^*(\mathcal{O}_{\mathcal{B}^{(1)}}(-\omega_1))) \cong R\Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(-p\omega_1))$. But $-p\omega_1 = s_1 s_2 \bullet ((p-3)\omega_2)$. Hence, by the Borel-Weil-Bott theorem (see [Jan03, II.5.5]), $\epsilon_0^{\mathcal{B}}(j_*\mathcal{O}_{\mathcal{B}^{(1)}}(-2\omega_1 - \omega_2)[2]) \cong \Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}((p-3)\omega_2))$. By [Jan03, II.5.6], it follows that $\epsilon_0^{\mathcal{B}}(j_*\mathcal{O}_{\mathcal{B}^{(1)}}(-2\omega_1 - \omega_2)[2]) \cong L((p-3)\omega_2)$.

Similar computations give the third line.

Now we consider the fourth line. For simplicity we write π_1 for $(\pi_1)^{(1)}$. We have, again by Lemma 1.4.1, $\epsilon_0^{\mathcal{B}}\big(j_*(\pi_1^*(\Omega^1_{(\mathbb{P}^2)^{(1)}})(-\omega_2))[1]\big) \cong R\Gamma\big(\mathcal{B}, \mathrm{Fr}^*(\pi_1^*(\Omega^1_{(\mathbb{P}^2)^{(1)}})(\omega_1))\big)[1]$. Using the exact sequence (3.1.1) we obtain a distinguished triangle

$$R\Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}})^{\oplus 3} \to R\Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(p\omega_1)) \to \epsilon_0^{\mathcal{B}}\big(j_*(\pi_1^*(\Omega^1_{(\mathbb{P}^2)^{(1)}})(-\omega_2))[1]\big).$$

Here the first arrow is the inclusion of G-modules $L(\omega_1)^{(1)} \hookrightarrow \mathrm{Ind}_B^G(p\omega_1)$. Hence we obtain $\epsilon_0^{\mathcal{B}}\big(j_*(\pi_1^*(\Omega^1_{(\mathbb{P}^2)^{(1)}})(-\omega_2))[1]\big) \cong L((p-2)\omega_1 + \omega_2)$. The claim for $L(\omega_1 + (p-2)\omega_2)$ follows by applying the outer automorphism of $\mathfrak{sl}(3)$.

Finally, the last irreducible module $L((p-2)\rho)$ is a quotient of the Weyl module $\big(\mathrm{Ind}_B^G((p-2)\rho)\big)^*$. More precisely, we have a short exact sequence

$$0 \to \Bbbk \to \big(\mathrm{Ind}_B^G((p-2)\rho)\big)^* \to L((p-2)\rho) \to 0.$$

Applying $(\epsilon_0^{\mathcal{B}})^{-1}$, and setting $\mathcal{L} := (\epsilon_0^{\mathcal{B}})^{-1} L((p-2)\rho)$, we get a distinguished triangle

$$j_*\mathcal{O}_{\mathcal{B}^{(1)}}(-\rho) \to j_*\mathcal{O}_{\mathcal{B}^{(1)}}(-2\rho)[3] \to \mathcal{L},$$

where we used the fact that

$$\epsilon_0^{\mathcal{B}}(j_*\mathcal{O}_{\mathcal{B}^{(1)}}(-2\rho)) \cong R\Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(-p\rho)) \cong \big(\mathrm{Ind}_B^G((p-2)\rho)\big)^*[-3]$$

by Lemma 1.4.1, Serre duality and Kempf's vanishing theorem ([Jan03, II.4.5]). Since $\mathrm{Hom}(\Bbbk, \big(\mathrm{Ind}_B^G((p-2)\rho)\big)^*)$ is one dimensional, we see that the first arrow in this triangle is the unique (up to a constant) non-zero map between the two objects. □

Remark 3.2.2. We have just shown, using equivalence $\epsilon_0^{\mathcal{B}}$, that

$$\mathrm{Ext}^3_{\widetilde{\mathcal{N}}}(j_*\mathcal{O}_{\mathcal{B}}, j_*\mathcal{O}_{\mathcal{B}}(-\rho))$$

is one dimensional (here, in fact, j should be replaced by the inclusion $\mathcal{B} \hookrightarrow \widetilde{\mathcal{N}}$ without Frobenius twists). One can compute this Ext group more directly: using the Koszul resolution of $\mathcal{O}_{\mathcal{B}}$ over $\mathrm{S}_{\mathcal{O}_{\mathcal{B}}}(\mathcal{T}_{\mathcal{B}})$ one can identify it with

$$H^3(\mathcal{O}_{\mathcal{B}}(-\rho)) \oplus H^2(\Omega^1_{\mathcal{B}}(-\rho)) \oplus H^1(\Omega^2_{\mathcal{B}}(-\rho)) \oplus H^0(\Omega^3_{\mathcal{B}}(-\rho)).$$

Here $\mathcal{T}_{\mathcal{B}}$ is the tangent sheaf to $\mathcal{B}$. Clearly, $H^3(\mathcal{O}_{\mathcal{B}}(-\rho))$ and $H^0(\Omega^3_{\mathcal{B}}(-\rho))$ vanish. By a result of Kumar-Lauritzen-Thomsen (see [BK04, Theorem 5.2.9]), $H^1(\Omega^2_{\mathcal{B}}(-\rho))$ also vanish[4], while $H^2(\Omega^1_{\mathcal{B}}(-\rho)) \cong \Bbbk$: by Serre duality the last claim is equivalent to $H^1(\mathcal{T}_{\mathcal{B}}(-\rho)) \cong \Bbbk$, which is checked below.

3.3 Projective covers

We define the objects *representing the projective covers* as in 2.3.

Proposition 3.3.1. *The following objects represent the projective covers of the simple objects of* $\mathrm{Mod}^{\mathrm{fg}}_0((\mathcal{U}\mathfrak{g})^0)$*:*

0	$\mathcal{P}$
$(p-3)\omega_2$	$p^*\big(((\pi_2^{(1)})^*\Omega^1_{(\mathbb{P}^2)^{(1)}})(\omega_2)\big)$
$(p-3)\omega_1$	$p^*\big(((\pi_1^{(1)})^*\Omega^1_{(\mathbb{P}^2)^{(1)}})(\omega_1)\big)$
$(p-2)\omega_1+\omega_2$	$\mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}(-\omega_2)$
$\omega_1+(p-2)\omega_2$	$\mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}(-\omega_1)$
$(p-2)\rho$	$\mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}$

where $\mathcal{P}$ is the non-trivial extension of $\mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}$ by $\mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}(-\rho)$ given by a non-zero element in the one dimensional space $H^1(\mathcal{T}_{\mathcal{B}}(-\rho)) \subset H^1(\mathcal{O}_{\widetilde{\mathcal{N}}}(-\rho))$.

Proof. For simplicity, in this proof we do not write the Frobenius twist $^{(1)}$. It should appear on every variety.

Let us begin with $\mathcal{O}_{\widetilde{\mathcal{N}}}$. We have

$$\mathrm{Ext}^*_{\widetilde{\mathcal{N}}}(\mathcal{O}_{\widetilde{\mathcal{N}}}, j_*(\mathcal{O}_{\mathcal{B}}(-\rho))) \cong \mathrm{Ext}^*_{\mathcal{B}}(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}}(-\rho)) \cong H^*(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(-\rho)) = 0$$

by adjunction and Borel-Weil-Bott theorem. Similar computations give the result for $j_*(\mathcal{O}_{\mathcal{B}}(-2\omega_1-\omega_2))[2]$ and $j_*(\mathcal{O}_{\mathcal{B}}(-\omega_1-2\omega_2))[2]$. The sequence (3.1.1) implies that

$$\mathrm{Ext}^*_{\widetilde{\mathcal{N}}}(\mathcal{O}_{\widetilde{\mathcal{N}}}, j_*\big(\pi_1^*\Omega^1_{\mathbb{P}^2}(-\omega_2)\big)[1]) \cong \mathrm{Ext}^*_{\mathcal{B}}(\mathcal{O}_{\mathcal{B}}, \pi_1^*\Omega^1_{\mathbb{P}^2}(-\omega_2)[1]) = 0.$$

[4]This can also be checked directly using the exact sequence $0 \to \mathcal{O}_{\mathcal{B}}(-\alpha_1-2\alpha_2) \oplus \mathcal{O}_{\mathcal{B}}(-2\alpha_1-\alpha_2) \to \Omega^2_{\mathcal{B}} \to \mathcal{O}_{\mathcal{B}}(-\rho) \to 0$.

The computation for the fifth simple object is similar. Finally, using the distinguished triangle from the definition of $\mathcal{L}$ and Serre duality we get $\mathrm{Ext}^*_{\widetilde{\mathcal{N}}}(\mathcal{O}_{\widetilde{\mathcal{N}}}, \mathcal{L}) = \Bbbk$.

The cases of $\mathcal{O}_{\widetilde{\mathcal{N}}}(-\omega_i)$ $(i = 1, 2)$ are similar.

Now let us consider $p^*((\pi_1^*\Omega^1_{\mathbb{P}^2})(\omega_1))$. The exact sequence (3.1.1) implies

$$\mathrm{Ext}^*_{\widetilde{\mathcal{N}}}(p^*(\pi_1^*\Omega^1_{\mathbb{P}^2} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{O}_\mathcal{B}(\omega_1), j_*\mathcal{O}_\mathcal{B}(-\rho)) \cong \mathrm{Ext}^*_\mathcal{B}(\pi_1^*\Omega^1_{\mathbb{P}^2}(\omega_1), \mathcal{O}_\mathcal{B}(-\rho)) = 0.$$

Here we have used that $-2\omega_1 - \omega_2 = w_0 \bullet (-\omega_1)$, and Borel-Weil-Bott theorem. The computations for the second to fifth simples are similar. For $\mathcal{L}$ we use its defining triangle. We have $\mathrm{Ext}^*_\mathcal{B}((\pi_1^*\Omega^1_{\mathbb{P}^2})(\omega_1), \mathcal{O}_\mathcal{B}(-\rho)) = 0$, and in computing $\mathrm{Ext}^*_\mathcal{B}((\pi_1^*\Omega^1_{\mathbb{P}^2})(\omega_1), \mathcal{O}_\mathcal{B}(-2\rho)[3])$, two non-zero modules appear in degree 0: $\big(H^3(\mathcal{O}_\mathcal{B}(-2\rho))\big)^{\oplus 3}$ and $\mathrm{Ind}_B^G(\omega_1)$. The map between these two modules is an isomorphism as in the proof of Proposition 3.2.1, hence $\mathrm{Ext}^*_{\widetilde{\mathcal{N}}}(p^*((\pi_1^*\Omega^1_{\mathbb{P}^2})(\omega_1)), \mathcal{L}) = 0$.

The computations for $p^*((\pi_2^*\Omega^1_{\mathbb{P}^2})(\omega_2))$ are similar.

We claim that $H^1(\mathcal{T}_\mathcal{B}(-\rho)) \cong \Bbbk$. This follows by the Borel-Weil-Bott theorem from the exact sequence

$$0 \to \mathcal{O}_\mathcal{B}(\alpha_1) \to \mathcal{T}_\mathcal{B} \to \pi_2^*(\mathcal{T}_{\mathbb{P}^2}) \to 0,$$

and vanishing of $R\Gamma(\pi_2^*(\mathcal{T}_{\mathbb{P}^2})(-\rho))$ (see *e.g.* [Dem76]). Thus we have the line $H^1(\mathcal{T}_\mathcal{B}(-\rho)) \subset H^1(\mathrm{S}(\mathcal{T}_\mathcal{B})(-\rho)) = \mathrm{Ext}^1_{\widetilde{\mathcal{N}}}(\mathcal{O}_{\widetilde{\mathcal{N}}}, \mathcal{O}_{\widetilde{\mathcal{N}}}(-\rho))$, which defines a triangle

$$\mathcal{O}_{\widetilde{\mathcal{N}}}(-\rho) \to \mathcal{P} \to \mathcal{O}_{\widetilde{\mathcal{N}}}.$$

Standard calculations give the result for $\mathcal{P}$ and the first three irreducible modules. The triangle defining $\mathcal{P}$ implies that we have $\mathrm{Ext}^*_{\widetilde{\mathcal{N}}}(\mathcal{P}, j_*\big(\pi_1^*\Omega^1_{\mathbb{P}^2}(\omega_1)\big)[1]) = H^*(\pi_1^*\Omega^1_{\mathbb{P}^2}(\omega_1))[1]$. Using (3.1.1), we have an exact sequence

$$0 \to H^0(\pi_1^*\Omega^1_{\mathbb{P}^2}(\omega_1)) \to \Bbbk^3 \to \mathrm{Ind}_B^G(\omega_1) \to H^1(\pi_1^*\Omega^1_{\mathbb{P}^2}(\omega_1)) \to 0$$

with invertible middle arrow (the other cohomology modules vanish). This proves the desired vanishing.

Finally, let us show that $\mathrm{Ext}^*_{\widetilde{\mathcal{N}}}(\mathcal{P}, \mathcal{L}) = 0$. First,

$$R\mathrm{Hom}_{\widetilde{\mathcal{N}}}(\mathcal{P}, j_*\mathcal{O}_\mathcal{B}(-\rho)) \cong R\Gamma(\mathcal{O}_\mathcal{B}) \cong \Bbbk,$$

and

$$R\mathrm{Hom}_{\widetilde{\mathcal{N}}}(\mathcal{P}, j_*\mathcal{O}_\mathcal{B}(-2\rho)[3]) \cong R\Gamma(\mathcal{O}_\mathcal{B}(-2\rho)[3]) \cong \Bbbk,$$

thus we only need to check that for nonzero morphisms

$$b : j_*\mathcal{O}_\mathcal{B}(-\rho) \to j_*\mathcal{O}_\mathcal{B}(-2\rho)[3], \quad \phi : \mathcal{P} \to j_*\mathcal{O}_\mathcal{B}(-\rho)$$

we have $b \circ \phi \neq 0$. It is clear from Remark 3.2.2 that $b = j_*(\beta) \circ \delta$, where $\delta : j_*\mathcal{O}_\mathcal{B}(-\rho) \to j_*\mathcal{T}_\mathcal{B}(-\rho)[1]$ is the shift by $-\rho$ of the class of the extension $0 \to j_*\mathcal{T}_\mathcal{B} \to \mathcal{O}_{\widetilde{\mathcal{N}}}/\mathcal{J}^2_\mathcal{B} \to j_*\mathcal{O}_\mathcal{B} \to 0$, and $\beta : \mathcal{T}_\mathcal{B}(-\rho)[1] \to \mathcal{O}_\mathcal{B}(-2\rho)[3]$ is a non-zero morphism; here $\mathcal{J}_\mathcal{B}$ is the ideal sheaf of the zero-section in $\widetilde{\mathcal{N}}$.

We claim that $\delta \circ \phi = j_*(\gamma) \circ \psi$, where $\psi : \mathcal{P} \twoheadrightarrow j_*\mathcal{O}_{\mathcal{B}}$ and $\gamma : \mathcal{O}_{\mathcal{B}} \to \mathcal{T}_{\mathcal{B}}(-\rho)[1]$ are nonzero morphisms. This follows from the definition of $\mathcal{P}$, which implies that $\mathcal{P}$ has a quotient, which is an extension of $j_*\mathcal{O}_{\mathcal{B}}(-\rho) \oplus j_*\mathcal{O}_{\mathcal{B}}$ by $j_*\mathcal{T}_{\mathcal{B}}(-\rho)$, such that the corresponding class in $\mathrm{Ext}^1_{\widetilde{\mathcal{N}}}(j_*\mathcal{O}_{\mathcal{B}}(-\rho), j_*(\mathcal{T}_{\mathcal{B}}(-\rho)))$ equals δ, while the corresponding class in $\mathrm{Ext}^1_{\widetilde{\mathcal{N}}}(j_*\mathcal{O}_{\mathcal{B}}, j_*(\mathcal{T}_{\mathcal{B}}(-\rho)))$ is non-trivial and is an image under j_* of an extension of coherent sheaves on $\mathcal{B}$.

It remains to show that the composition $j_*\beta \circ j_*\gamma \circ \psi$ is nonzero. The composition $\beta \circ \gamma \in \mathrm{Ext}^3_{\mathcal{B}}(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}}(-2\rho)) = H^3(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(-2\rho)) = \Bbbk$ is nonzero, because it coincides with the Serre duality pairing of nonzero elements β, γ in the two dual one-dimensional spaces $H^1(\mathcal{T}_{\mathcal{B}}(-\rho))$, $H^2(\mathcal{T}^*_{\mathcal{B}}(-\rho))$. Consequently, the composition $j_*(\beta \circ \gamma) \circ \psi$ is also nonzero, since under the isomorphism

$$\mathrm{Hom}(\mathcal{P}, j_*\mathcal{O}_{\mathcal{B}}(-2\rho)[3]) \cong \mathrm{Hom}(j^*\mathcal{P}, \mathcal{O}_{\mathcal{B}}(-2\rho)[3]) \cong \mathrm{Hom}(\mathcal{O}_{\mathcal{B}}(-\rho) \oplus \mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}}(-2\rho)[3])$$

it corresponds to the composition of $\beta \circ \gamma$ and projection to the second summand. $\square$

Chapter II

Geometric braid group action

In this section we construct and study an action of the extended affine braid group B'_{aff} on the categories $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$ and $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$. This result will be used in chapter III, but is also interesting in its own right.

Sections 1 to 7 of this chapter were published in [Ric08a]. In section 8 we present an alternate (and more general) proof of the main step of the construction.

Sections 1 and 8 are joint works with Roman Bezrukavnikov.

Introduction

0.1

Let G be, as in chapter I, a connected, semi-simple, simply-connected algebraic group over an algebraically closed field $\Bbbk$ of characteristic p, and let $\mathfrak{g} = \mathrm{Lie}(G)$. In [BMR06], Bezrukavnikov, Mirković and Rumynin have constructed an action of the extended affine braid group associated with G on the category $\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}_\chi^{(1)}}(\widetilde{\mathfrak{g}}^{(1)})$, when p is greater than the Coxeter number h of G (here $\chi \in \mathfrak{g}^*$ is nilpotent, and $\mathcal{B}_\chi$ is the corresponding Springer fiber). Their construction relies on deep results relating the modules over $\mathcal{U}\mathfrak{g}$ (the enveloping algebra of $\mathfrak{g}$), $\mathcal{D}$-modules on the flag variety of G, and coherent sheaves on $\widetilde{\mathfrak{g}}^{(1)}$ (see I.1.2).

In this chapter we show that this action can be defined geometrically, without any reference to Representation Theory. In particular, we obtain that the action can also be defined for other characteristics, including 0. We also obtain that similar actions can be defined on various other categories, such as $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$, $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$, $\mathcal{D}^b\mathrm{Coh}^G(\widetilde{\mathfrak{g}})$ or $\mathcal{D}^b\mathrm{Coh}^G(\widetilde{\mathcal{N}})$.

For $\Bbbk = \mathbb{C}$, this action is related to Kazhdan-Lusztig's and Ginzburg's interpretation of the equivariant K-theory of the Steinberg variety, and to Springer representations of the Weyl group on the homology of Springer fibers.

0.2

We use the notations of I.1.1 and I.1.2. The extended affine Weyl group $W'_{\mathrm{aff}} := W \ltimes \mathbb{X}$ has a natural "length function" ℓ, although it is not a Coxeter group in general (see 1.1). The extended affine braid group B'_{aff} is by definition the group with presentation:

$$\begin{array}{ll} \text{Generators:} & T_w \ (w \in W'_{\mathrm{aff}}); \\ \text{Relations:} & T_v T_w = T_{vw} \text{ if } \ell(vw) = \ell(v) + \ell(w). \end{array}$$

This definition is similar to the "Iwahori-Matsumoto presentation" of the corresponding Hecke algebra $\mathcal{H}'_{\mathrm{aff}}$. If $x \in \mathbb{X}$, write $x = x_1 - x_2$ with x_1, x_2 dominant weights. Then $\theta_x := T_{x_1}(T_{x_2})^{-1}$ depends only on x. If $\alpha, \beta \in \Phi$, we denote by $n_{\alpha,\beta}$ the order of $s_\alpha s_\beta$ in W. Our first step[1] (see section 1), is a second presentation of B'_{aff}, which is an analogue of the "Bernstein presentation" of $\mathcal{H}'_{\mathrm{aff}}$. The idea of this presentation is due to Lusztig (see *e.g.* [Lus89]). It is given by:

$$\begin{array}{ll} \text{Generators:} & T_{s_\alpha} \ (\alpha \in \Phi), \ \theta_x \ (x \in \mathbb{X}); \\ \text{Relations:} & (1) \ T_{s_\alpha} T_{s_\beta} \cdots = T_{s_\beta} T_{s_\alpha} \cdots \ (n_{\alpha,\beta} \text{ elements on each side}); \\ & (2) \ \theta_x \theta_y = \theta_{x+y}; \\ & (3) \ T_{s_\alpha} \theta_x = \theta_x T_{s_\alpha} \text{ if } \langle x, \alpha^\vee \rangle = 0; \\ & (4) \ \theta_x = T_{s_\alpha} \theta_{x-\alpha} T_{s_\alpha} \text{ if } \langle x, \alpha^\vee \rangle = 1. \end{array}$$

Our main result is the construction of a *weak*[2] action of B'_{aff} on the category $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$, by convolution. Using the preceding presentation, to construct this action it is sufficient to define kernels associated to the generators T_{s_α} and θ_x, and to check relations (1) to (4) for these kernels. The kernel associated with T_{s_α} is $\mathcal{O}_{S_\alpha}$ for some closed subvariety $S_\alpha \subset \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$ (see subsection 2.3 for a precise definition), and the kernel associated with θ_x is $\Delta_* \mathcal{O}_{\widetilde{\mathfrak{g}}}(x)$ where $\Delta : \widetilde{\mathfrak{g}} \hookrightarrow \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$ is the diagonal embedding. Relations (2), (3) and (4) for these kernels are easy to prove.

The most difficult relations to prove are the "finite braid relations", *i.e.* relations (1). We give two proofs of these relations.

For the first proof we have to assume that G has no factor of type $\mathbf{G}_2$ and that $p \neq 2$ if R is not simply-laced, and to perform a case-by-case analysis, depending on whether α and β generate a root system of type $\mathbf{A}_1 \times \mathbf{A}_1$, $\mathbf{A}_2$ or $\mathbf{B}_2$ (see sections 3 and 4). Our proof involves the study of Demazure-like "resolutions"[3] $\widetilde{\mathcal{Z}}_{(s_1,s_2,\cdots,s_n)} \to S_w$. Here w is the element of W corresponding to the finite braid relation under consideration[4], S_w is a subvariety of the product of $\mathfrak{g}^*$ with the G-orbit closure $\mathcal{X}_w \subset (G/B) \times (G/B)$ associated with w, and $\widetilde{\mathcal{Z}}_{(s_1,s_2,\cdots,s_n)}$ is a subvariety of the product of $\mathfrak{g}^*$ with the Demazure resolution

[1] After the paper [Ric08a] was submitted, Valerio Toledano Laredo pointed out to us that this presentation is also proved in Macdonald's book [Mac03]. Our proof is different.

[2] See subsection 2.2.

[3] These are not really resolutions of singularities, as the variety $\widetilde{\mathcal{Z}}_{(s_1,s_2,\cdots,s_n)}$ is singular in general. But we show that they share some properties with resolutions of singularities.

[4] *I.e.* w is the longest element of the Weyl group of the standard parabolic subgroup of G associated with $\{\alpha, \beta\}$.

of $\mathcal{X}_w$ associated with the reduced decomposition $w = s_1 s_2 \cdots s_n$. Moreover, the morphism $\widetilde{\mathcal{Z}}_{(s_1,s_2,\cdots,s_n)} \to S_w$ is induced by the morphism from the Demazure resolution to $\mathcal{X}_w$.

The second proof of the relations is given in section 8. Here we do not make any assumption on the group, but we assume that p is very good for G. The main ideas of this second proof come from [Bez06a].

Finally, we obtain (see Theorem 2.3.2) that if either G has no factor of type $\mathbf{G}_2$ and $\mathrm{char}(\Bbbk) \neq 2$ if R is not simply-laced, or if p is very good for G, there exists an action of B'_{aff} on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$ such that

(i) The action of θ_x is given by the convolution with kernel $\Delta_* \mathcal{O}_{\widetilde{\mathfrak{g}}}(x)$;
(ii) The action of T_{s_α} is given by the convolution with kernel $\mathcal{O}_{S_\alpha}$.

In sections 5 to 7 we study the compatibility of this action with the inclusion $\widetilde{\mathcal{N}} \hookrightarrow \widetilde{\mathfrak{g}}$, and with some representation-theoretic constructions.

First, in section 5 we show that one can similarly define an action of B'_{aff} on the category $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$, such that the following diagram is commutative for any $b \in B'_{\mathrm{aff}}$, where $i : \widetilde{\mathcal{N}} \hookrightarrow \widetilde{\mathfrak{g}}$ denotes the natural embedding:

$$\begin{tikzcd}
\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}) \arrow[r, "i_*"] \arrow[d, "b"] & \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}) \arrow[d, "b"] \\
\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}) \arrow[r, "i_*"] & \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}).
\end{tikzcd}$$

In section 6 we show that, for $p > h$, the action of B'_{aff} on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$, or rather the similar action on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})$ (the supscript ${}^{(1)}$ denotes the Frobenius twist), extends the action on $\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}_\chi^{(1)}}(\widetilde{\mathfrak{g}}^{(1)})$ considered in [BMR06]. Hence, as a consequence of our results in section 5, the action by intertwining functors on $\mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,\chi)}(\mathcal{U}\mathfrak{g})$ of [BMR06] factors through an action on $\mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{\chi}((\mathcal{U}\mathfrak{g})^\lambda)$ (see I.1.2 for notations).

Finally, in section 7 we explain the relation between our results for $\Bbbk = \mathbb{C}$ and some classical constructions. In particular, the action on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$ gives a categorical framework for Ginzburg's isomorphism between the equivariant K-theory of the Steinberg variety and the extended affine Hecke algebra $\mathcal{H}'_{\mathrm{aff}}$, and for Lusztig's construction of irreducible $\mathcal{H}'_{\mathrm{aff}}$-modules over $\mathbb{C}$. Also, the induced action on the Grothendieck group of $\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}_\chi}(\widetilde{\mathcal{N}})$ gives Springer representations of W on the homology of $\mathcal{B}_\chi$.

0.3

To finish this introduction, let us say a few words on the importance of this braid group action. First, its importance was emphasized in Bezrukavnikov's talk at ICM 2006: this action "encodes" the *exotic* t-structure on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$ and $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$. In positive characteristic, this t-structure comes from the equivalence with representations of $\mathcal{U}\mathfrak{g}$. It also has an interesting interpretation in characteristic zero (see [Bez06b] for details). In fact, our construction will be a step in the proof, by Bezrukavnikov and Mirković, of Lusztig's conjecture relating irreducible $\mathcal{U}\mathfrak{g}$-modules to elements of the canonical basis in the Borel-Moore homology of a Springer fiber ([Lus98], [Lus99]). Similar actions also appear in Gukov and Witten's work on gauge theory and geometric Langlands program (see [GW06]), and in Bridgeland's study of stability conditions on triangulated categories (see [Bez06b] for details on this point). Finally, we will use this construction to study a certain Koszul duality for modular representations of $\mathfrak{g}$ (see chapter III or [Ric08b]).

1 A Bernstein-type presentation of the braid group

1.1 Statement of the theorem

Let us introduce some notations concerning Weyl groups and braid groups. Recall the notations of I.1.1. We denote by $t_x \in W'_{\mathrm{aff}}$ the translation corresponding to $x \in \mathbb{X}$. Let $S := \{s_\alpha,\ \alpha \in \Phi\}$ be the usual set of generators of W. Let also $S_{\mathrm{aff}} \subset W_{\mathrm{aff}}$ be the usual set of generators of W_{aff}; that is, S_{aff} contains S together with additional reflections corresponding to the highest coroot of each irreducible component of R. Then (W, S) and $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ are Coxeter systems. We denote by ℓ their length function.

Let $A_0 = \{\lambda \in \mathbb{X} \otimes_{\mathbb{Z}} \mathbb{R} \mid \forall \alpha \in R^+, 0 < \langle \lambda, \alpha^\vee \rangle < 1\}$ be the fundamental alcove. If $\Omega \subset W'_{\mathrm{aff}}$ is the stabilizer of A_0 for the standard action on $\mathbb{X} \otimes_{\mathbb{Z}} \mathbb{R}$, we have $W'_{\mathrm{aff}} \cong W_{\mathrm{aff}} \rtimes \Omega$. We can use this isomorphism to extend ℓ to W'_{aff}, setting $\ell(\omega) = 0$ for $\omega \in \Omega$. Then, for $w \in W$ and $x \in \mathbb{X}$ ([IM65, prop. 1.23]):

$$\ell(w \cdot t_x) \;=\; \sum_{\substack{\alpha \in R^+, \\ w\alpha \in R^+}} |\langle x, \alpha^\vee \rangle| \;+\; \sum_{\substack{\alpha \in R^+, \\ w\alpha \in R^-}} |1 + \langle x, \alpha^\vee \rangle|. \tag{1.1.1}$$

Now, let us recall the definition of the braid group associated with a Coxeter group H, with length ℓ_H. By definition, the braid group B_H is the group with generators the $\{T_v,\ v \in H\}$ and relations $T_{uv} = T_u T_v$ if $\ell_H(uv) = \ell_H(u) + \ell_H(v)$. In particular we have the braid group B_0 associated with W, and the affine braid group B_{aff} associated with W_{aff}. The group W'_{aff} is not a Coxeter group, but we have defined a length function ℓ on it. Hence we can use the same recipe to define the extended affine braid group B'_{aff}. There are natural inclusions

$$B_0 \subset B_{\mathrm{aff}} \subset B'_{\mathrm{aff}}.$$

Moreover, there is a natural isomorphism $B'_{\mathrm{aff}} \cong B_{\mathrm{aff}} \rtimes \Omega$.

There is a canonical section $C : W'_{\mathrm{aff}} \to B'_{\mathrm{aff}}$ (which sends W_{aff} into B_{aff} and W into B_0) of the canonical morphism $B'_{\mathrm{aff}} \to W'_{\mathrm{aff}}$, defined by $C(w) := T_w$ (this is not a group morphism). From now on we will not use the notation T_w anymore, except when $w = s_\alpha \in S$; moreover, in this case, we will simplify T_{s_α} in T_α. We denote by $n_{\alpha,\beta}$ the order of $s_\alpha s_\beta$ in W, for $\alpha, \beta \in \Phi$.

If λ and μ are dominant weights, $\ell(t_\lambda t_\mu) = \ell(t_\lambda) + \ell(t_\mu)$, see (1.1.1). Hence

$$C(t_\lambda t_\mu) = C(t_\lambda) C(t_\mu). \tag{1.1.2}$$

Let $x \in \mathbb{X}$. We write $x = x_1 - x_2$ with x_1 and x_2 dominant weights. Then we set $\theta_x := C(t_{x_1})C(t_{x_2})^{-1}$. This does not depend on the chosen decomposition, due to formula (1.1.2). In this section we prove:

Theorem 1.1.3. *B'_{aff} admits a presentation with generators $\{T_\alpha,\ \alpha \in \Phi\} \cup \{\theta_x,\ x \in \mathbb{X}\}$ and relations:*

1. *$T_\alpha T_\beta \cdots = T_\beta T_\alpha \cdots$ ($n_{\alpha,\beta}$ elements on each side);*
2. *$\theta_x \theta_y = \theta_{x+y}$;*
3. *$T_\alpha \theta_x = \theta_x T_\alpha$ if $\langle x, \alpha^\vee \rangle = 0$, i.e. $s_\alpha(x) = x$;*
4. *$\theta_x = T_\alpha \theta_{x-\alpha} T_\alpha$ if $\langle x, \alpha^\vee \rangle = 1$, i.e. $s_\alpha(x) = x - \alpha$.*

This theorem is an analogue of the well known result of J. Bernstein concerning the corresponding Hecke algebra. Relations 1 are called "finite braid relations" in the sequel.

The facts that the elements T_α and θ_x generate B'_{aff}, and satisfy the relations of the theorem, are proved in [Lus89, 2.7, 2.8]. We denote by $\hat{B}$ the group with the given presentation. There exists a (surjective[5]) morphism

$$\psi : \hat{B} \twoheadrightarrow B'_{\mathrm{aff}}.$$

To prove the theorem we construct an inverse ϕ to this morphism. To avoid confusion, in this section we denote by $\hat{T}_\alpha$ and $\hat{\theta}_x$ the images of the generators in $\hat{B}$. Hence we have $\psi(\hat{T}_\alpha) = T_\alpha$, $\psi(\hat{\theta}_x) = \theta_x$.

1.2 A second "length function"

In this subsection we introduce a second "length function" on W'_{aff}, denoted L, with values in $\mathbb{Z}$. Let $\mathscr{H}$ by the set of reflection hyperplanes of W_{aff} in $\mathbb{X} \otimes \mathbb{R}$, and $\mathscr{A}$ be the set of alcoves. Let C^0 be the fundamental chamber, *i.e.*

$$C^0 = \{x \in \mathbb{X} \otimes \mathbb{R} \mid \forall \alpha \in \Phi, \langle x, \alpha^\vee \rangle \geq 0\}.$$

If $H \in \mathscr{H}$, we denote by E_H^+ the half space defined by H that intersects all translates of C^0, and by E_H^- the other half space. Then, following Jantzen and Lusztig (see [Lus80a]) we introduce the function d on $\mathscr{A}^2$, defined by

$$d(A,B) = \#\{H \in \mathscr{H} \mid A \subset E_H^- \text{ and } B \subset E_H^+\} \ - \ \#\{H \in \mathscr{H} \mid A \subset E_H^+ \text{ and } B \subset E_H^-\}.$$

It is clear from the definition that $d(A,B) = -d(B,A)$. Moreover, d satisfies the following formula for three alcoves A, B and C (see [Lus80a, 1.4.1]):

$$d(A,B) + d(B,C) + d(C,A) = 0. \tag{1.2.1}$$

[5] We do not use this surjectivity in our proof, but rather re-prove it.

Now we can define the function L on W'_{aff} by setting

$$L(w) := d(A_0, w^{-1}A_0)$$

(recall that A_0 denotes the fundamental alcove). For $w \in W$ we have $L(w) = -\ell(w)$, and for $x \in \mathbb{X}$ antidominant we have $L(t_x) = \ell(t_x)$. Similarly, if x is dominant we have $L(t_x) = -\ell(t_x)$. Moreover,

$$|L(w)| \leq \ell(w)$$

for any $w \in W'_{\mathrm{aff}}$ (for all of this, use [Hum90, 4.5]).

Lemma 1.2.2. *For any $u, w \in W'_{\mathrm{aff}}$, we have $|L(wu) - L(u)| \leq \ell(w)$. Moreover, for any $w \in W'_{\mathrm{aff}}$ there exists $u \in W'_{\mathrm{aff}}$ such that $L(wu) - L(u) = -\ell(w)$.*

Proof. Using formula (1.2.1) we have

$$L(wu) - L(u) = d(A_0, u^{-1}w^{-1}A_0) - d(A_0, u^{-1}A_0) = d(u^{-1}A_0, u^{-1}w^{-1}A_0).$$

Hence $|L(wu) - L(u)|$ is at most the number of hyperplanes in $\mathscr{H}$ separating $u^{-1}A_0$ and $u^{-1}w^{-1}A_0$, which equals the number of hyperplanes separating A_0 and $w^{-1}A_0$. This number is precisely $\ell(w^{-1}) = \ell(w)$.

Let us now consider the second assertion. Let ξ be a point in A_0. Let $u \in W$ be such that $u^{-1}(w^{-1}(\xi) - \xi)$ is in $w_0 C^0$, where w_0 is the longest element of W. Then it is clear that $d(u^{-1}A_0, u^{-1}w^{-1}A_0) = -\ell(w)$. □

1.3 Computations in W'_{aff}

In 1.1 we have defined a section C of the morphism $B'_{\mathrm{aff}} \to W'_{\mathrm{aff}}$. Now, let us define another section $S : W'_{\mathrm{aff}} \to B'_{\mathrm{aff}}$ by setting $S(w_f \cdot t_x) := C(w_f)\theta_x$ for $w_f \in W$ and $x \in \mathbb{X}$, where we have used the isomorphism $W'_{\mathrm{aff}} \cong W \ltimes \mathbb{X}$. We will show that one can recover C from S.

Lemma 1.3.1. *Let $u, w \in W'_{\mathrm{aff}}$ be such that $L(wu) = L(u) - \ell(w)$. Then we have $S(wu) = C(w)S(u)$.*

Proof. First, let us remark that the hypothesis and the conclusion are invariant by replacing u by ut_x for some $x \in \mathbb{X}$. Hence we can assume that $u \in W$. We write $w = w_f t_\lambda$ for some $\lambda \in \mathbb{X}$, $w_f \in W$. Then

$$L(wu) - L(u) = d(u^{-1}A_0, u^{-1}w^{-1}A_0) = d(u^{-1}A_0, u^{-1}(w_f)^{-1}A_0 - u^{-1}(\lambda)).$$

As u and w_f are in W, and as every hyperplane H between $u^{-1}A_0$ and $u^{-1}w^{-1}A_0$ is crossed in the direction $E_H^+ \rightsquigarrow E_H^-$ we must have the inequality $\langle -u^{-1}(\lambda), \alpha^\vee \rangle \leq 1$ for any $\alpha \in R^+$, *i.e.* $\langle u^{-1}(\lambda), \alpha^\vee \rangle \geq -1$. Moreover, for any $\alpha \in R^+$ such that $w_f u(\alpha) \in R^+$ we have $\langle u^{-1}(\lambda), \alpha^\vee \rangle \geq 0$. Indeed, in this case $u^{-1}(w_f)^{-1}A_0$ is in $E_{H_\alpha}^+$ for H_α the reflection hyperplane of s_α. Hence if $\langle u^{-1}(\lambda), \alpha^\vee \rangle$ were -1 then to go from $u^{-1}A_0$ to $u^{-1}w^{-1}A_0$ we would have to cross the hyperplane

$$H := \{x \in \mathbb{X} \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle x, \alpha^\vee \rangle = 1\}$$

in the "wrong" direction (*i.e.* $E_H^- \rightsquigarrow E_H^+$).

Let us write $u^{-1}(\lambda) = \mu_1 - \mu_2$ with μ_1 and μ_2 dominant weights. We have $wu = (w_f)t_\lambda u = w_f u t_{u^{-1}(\lambda)}$. Hence $wut_{\mu_2} = w_f u t_{\mu_1}$. As μ_1 is dominant and $w_f u \in W$, $\ell(w_f u t_{\mu_1}) = \ell(w_f u) + \ell(t_{\mu_1})$ (see (1.1.1)). Hence $C(w_f u t_{\mu_1}) = C(w_f u)C(t_{\mu_1})$. We will now prove that, also, $\ell(wut_{\mu_2}) = \ell(wu) + \ell(t_{\mu_2})$. It will follow that $C(wut_{\mu_2}) = C(wu)C(t_{\mu_2})$, and finally that $S(wu) = C(wu)$.

So, let us prove that $\ell(wut_{\mu_2}) = \ell(wu) + \ell(t_{\mu_2})$. Using formula (1.1.1), we have

$$\begin{aligned}
\ell(t_{\mu_2}) &= \sum_{\alpha \in R^+} \langle \mu_2, \alpha^\vee \rangle, \\
\ell(wu) &= \sum_{\substack{\alpha \in R^+ \\ w_f u(\alpha) \in R^+}} |\langle \mu_1 - \mu_2, \alpha^\vee \rangle| \ + \sum_{\substack{\alpha \in R^+, \\ w_f u(\alpha) \in R^-}} |1 + \langle \mu_1 - \mu_2, \alpha^\vee \rangle|, \\
\ell(wut_{\mu_2}) &= \sum_{\substack{\alpha \in R^+, \\ w_f u(\alpha) \in R^+}} \langle \mu_1, \alpha^\vee \rangle \ + \sum_{\substack{\alpha \in R^+, \\ w_f u(\alpha) \in R^-}} (1 + \langle \mu_1, \alpha^\vee \rangle).
\end{aligned}$$

We know (see above) that for any $\alpha \in R^+$, $\langle u^{-1}(\lambda), \alpha^\vee \rangle \geq -1$, and, for any $\alpha \in R^+$ such that $w_f u(\alpha) \in R^+$, $\langle u^{-1}(\lambda), \alpha^\vee \rangle \geq 0$. The result easily follows.

Finally we have proved that $S(wu) = C(wu)$. By hypothesis $|L(wu)| = |L(u) - \ell(w)| = \ell(u) + \ell(w)$ (because u is in W). On the other hand we have the inequalities $|L(wu)| \leq \ell(wu) \leq \ell(w) + \ell(u)$. We deduce that we must have $\ell(wu) = \ell(w) + \ell(u)$. Hence $C(wu) = C(w)C(u) = C(w)S(u)$. This concludes the proof. □

1.4 Computations in $\hat{B}$

The braid group B_0 is well known to have a presentation with generators the T_α ($\alpha \in \Phi$) and relations (1) of Theorem 1.1.3. Hence there exists a group morphism $\sigma : B_0 \to \hat{B}$, which sends T_α to $\hat{T}_\alpha$. We define $C' := \sigma \circ C|_W : W \to \hat{B}$. Then we can define the lift

$$S' : W'_{\mathrm{aff}} \to \hat{B}$$

by setting $S'(w_f t_x) := C'(w_f)\hat{\theta}_x$ for $w_f \in W$, $x \in \mathbb{X}$. The following diagram is commutative:

$$\begin{tikzcd}
 & W'_{\mathrm{aff}} \arrow[dl, "S'"'] \arrow[dr, "S"] & \\
\hat{B} \arrow[rr, "\psi"'] & & B'_{\mathrm{aff}}
\end{tikzcd}$$

The next proposition is the key step in our proof of Theorem 1.1.3.

Proposition 1.4.1. *Let $w, u_1, u_2 \in W'_{\mathrm{aff}}$ such that $L(wu_1) = L(u_1) - \ell(w)$ and $L(wu_2) = L(u_2) - \ell(w)$. Then*

$$S'(wu_1)(S'(u_1))^{-1} = S'(wu_2)(S'(u_2))^{-1}.$$

Proof. We use induction on $\ell(w)$. Assume we know the result for v and w, and that $\ell(vw) = \ell(v) + \ell(w)$. Let u_1 and u_2 be as in the proposition, for vw instead of w. For $i = 1, 2$ we have $L(vwu_i) \geq L(wu_i) - \ell(v) \geq L(u_i) - \ell(w) - \ell(v)$ (by Lemma 1.2.2). As the two extreme terms are equal by assumption, we must have $L(vwu_i) = L(wu_i) - \ell(v)$ and $L(wu_i) = L(u_i) - \ell(w)$. Applying the result for v, wu_1, wu_2 and w, u_1, u_2 we obtain the result for vw, u_1, u_2. Hence we only have to prove the proposition for w of length 0 or 1. We also only have to prove it for $u_i \in W$ (use relation (2) and the definition of S'). Without loss of generality we can assume R is irreducible ($\hat{B}$ is the product of the subgroups corresponding to each irreducible component of R).

(i) First, consider the easiest case $w = s \in S$. For $i = 1, 2$ we have by definition $d(u_i^{-1}A_0, u_i^{-1}sA_0) = -1$. Hence, if $s = s_\alpha$, $u_i^{-1}(\alpha) \in R^+$. Then $\ell(su_i) = \ell(u_i) + 1$ (use the criterion provided by [Hum90, 1.6, 1.7]). Hence $S'(su_i) = C'(su_i) = C'(s)C'(u_i) = S'(s)S'(u_i)$. This proves the result in this case.

(ii) Next, assume w is in $S_{\text{aff}} - S$. Then $w = t_\gamma s_\gamma$ for γ the highest short root of R. We have to show that

$$S'(wu)(S'(u))^{-1} := C'(s_\gamma u)\hat{\theta}_{-u^{-1}(\gamma)}C'(u)^{-1}$$

doesn't depend on the choice of $u \in W$ such that $d(u^{-1}A_0, u^{-1}s_\gamma A_0 + u^{-1}(\gamma)) = -1$. This condition amounts to requiring $u^{-1}(\gamma) \in R^-$. In particular, w_0 fits (recall that w_0 denotes the longest element of W). By descending induction on $l(u)$, we will show that $C'(s_\gamma u)\hat{\theta}_{-u^{-1}(\gamma)}C'(u)^{-1} = C'(s_\gamma w_0)\hat{\theta}_{-w_0(\gamma)}C'(w_0)^{-1}$ for any $u \in W$ such that $u^{-1}(\gamma) \in R^-$.

Assume $u \neq w_0$. Then choose $\beta \in \Phi$ such that $\ell(us_\beta) = \ell(u) + 1$, *i.e.* $u(\beta) \in R^+$. Then $\beta \neq -u^{-1}(\gamma)$, hence $s_\beta u^{-1}(\gamma) \in R^-$, so that we can apply the induction hypothesis to us_β. Moreover,

$$C'(s_\gamma us_\beta)\hat{\theta}_{-s_\beta u^{-1}(\gamma)}C'(us_\beta)^{-1} = C'(s_\gamma us_\beta)\hat{\theta}_{-s_\beta u^{-1}(\gamma)}(\hat{T}_\beta)^{-1}C'(u)^{-1}.$$

As γ is a short root and a dominant weight, and $u(\beta)$ is a positive root, $\langle \gamma, u(\beta)^\vee \rangle = \langle u^{-1}(\gamma), \beta^\vee \rangle$ is 0 or 1. First, assume it is 0. Then $s_\beta u^{-1}(\gamma) = u^{-1}(\gamma)$, and by relation (3) we have $\hat{\theta}_{-u^{-1}(\gamma)}\hat{T}_\beta^{-1} = \hat{T}_\beta^{-1}\hat{\theta}_{-u^{-1}(\gamma)}$. Moreover, $s_\gamma u(\beta) = u(\beta) \in R^+$, hence $\ell(s_\gamma us_\beta) = \ell(s_\gamma u) + 1$, and then $C'(s_\gamma us_\beta) = C'(s_\gamma u)\hat{T}_\beta$. This concludes the proof in this case.

Now assume $\langle \gamma, u(\beta)^\vee \rangle = 1$. Then $s_\beta u^{-1}(\gamma) = u^{-1}(\gamma) - \beta$, and by relation (4) we have $\hat{\theta}_{-s_\beta u^{-1}(\gamma)} = \hat{T}_\beta\hat{\theta}_{-u^{-1}(\gamma)}\hat{T}_\beta$. Moreover, $s_\gamma u(\beta) \in R^-$ (as $\langle u(\beta), \gamma^\vee \rangle > 0$), hence $\ell(s_\gamma us_\beta) = \ell(s_\gamma u) - 1$. One concludes as before.

(iii) Finally, consider some w with $\ell(w) = 0$. Write $w = w_f t_\lambda$. Using formula (1.1.1) we have $\langle \lambda, \alpha^\vee \rangle = 0$ if $w_f(\alpha) \in R^+$, and $\langle \lambda, \alpha^\vee \rangle = -1$ if $w_f(\alpha) \in R^-$. There is no condition on u in this case. Hence we have to prove that $S'(wu)(S'(u))^{-1} = S'(w)$ for any $u \in W$. We will prove it by (ascending) induction on $\ell(u)$. If $u \neq \mathrm{Id}$, let $\beta \in \Phi$ and $v \in W$ be such that $u = vs_\beta$, with $l(v) = l(u) - 1$. Then $v(\beta) \in R^+$. We have $S'(wu)(S'(u))^{-1} = C'(w_f vs_\beta)\hat{\theta}_{s_\beta v^{-1}(\lambda)}(\hat{T}_\beta)^{-1}C'(v)^{-1}$.

First, assume $\ell(w_f v s_\beta) = \ell(w_f v) + 1$, *i.e.* $C'(w_f v s_\beta) = C'(w_f v)\hat{T}_\beta$. Then $w_f v(\beta) \in R^+$. Hence $\langle \lambda, v(\beta)^\vee \rangle = 0 = \langle v^{-1}(\lambda), \beta^\vee \rangle$. Hence $s_\beta v^{-1}(\lambda) = v^{-1}(\lambda)$, and relation (3) gives $\hat{T}_\beta \hat{\theta}_{v^{-1}(\lambda)} = \hat{\theta}_{v^{-1}(\lambda)} \hat{T}_\beta$. Then the result for u follows from the result for v.

Next, assume $\ell(w_f v s_\beta) = \ell(w_f v) - 1$, *i.e.* $C'(w_f v s_\beta) = C'(w_f v)(\hat{T}_\beta)^{-1}$. Then $w_f v(\beta) \in R^-$. Hence $\langle v^{-1}(\lambda), \beta^\vee \rangle = -1$. And the result for u follows from the result for v and relation (4) applied to $s_\beta v^{-1}(\lambda)$. □

1.5 End of the proof

We define a group morphism $\phi : B'_{\mathrm{aff}} \to \hat{B}$ by setting, for any $w \in W'_{\mathrm{aff}}$, $\phi(C(w)) = S'(wu)(S'(u))^{-1}$ for some $u \in W'_{\mathrm{aff}}$ such that $L(wu) = L(u) - \ell(w)$ (such a u exists by Lemma 1.2.2, and $S'(wu)(S'(u))^{-1}$ does not depend on the choice of u, due to Proposition 1.4.1). We have already proved that these elements satisfy the relations of the definition of B'_{aff} in the beginning of the proof of Proposition 1.4.1.

Recall that $\psi : \hat{B} \to B'_{\mathrm{aff}}$ denotes the canonical morphism. It follows from Lemma 1.3.1 and the diagram at the beginning of subsection 1.4 that $\psi \circ \phi = \mathrm{Id}$. If $s \in S$ then $L(s) = -\ell(s)$, hence one may take $u = 1$. Thus $\phi \circ \psi(\hat{T}_s) = \phi(T_s) = \hat{T}_s$. Similarly, if $x \in \mathbb{X}$ is dominant then $L(t_x) = -\ell(t_x)$. Hence $\phi \circ \psi(\hat{\theta}_x) = \phi(\theta_x) = \phi(C(t_x)) = \hat{\theta}_x$. As these elements generate $\hat{B}$ (use relation (2)), we conclude that $\phi \circ \psi = \mathrm{Id}$. This concludes the proof of Theorem 1.1.3.

2 Action of the braid group by convolution

2.1 Convolution

By a *variety* we mean a reduced, separated scheme of finite type over $\Bbbk$ (in particular, we do not assume it is irreducible). If X is a variety, we identify the derived category $\mathcal{D}^b\mathrm{Coh}(X)$ with the full subcategory of $\mathcal{D}^b\mathrm{QCoh}(X)$ whose objects have coherent cohomology sheaves ([BGI71, II.2.2.2.1]; see also [Bor87, VI.2.11] for a sketch of a more elementary proof, following P. Deligne).

If X is a scheme and $i : Z \hookrightarrow X$ a closed subscheme, for simplicity we sometimes write $\mathcal{O}_Z$ for $i_*\mathcal{O}_Z$. We will also sometimes write simply $(- \otimes_X -)$ for $(- \otimes_{\mathcal{O}_X} -)$, and similarly for the derived tensor product.

Let X, Y be varieties. We denote by $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ the projections. We define the full subcategory

$$\mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(X \times Y) \ \subset \ \mathcal{D}^b\mathrm{Coh}(X \times Y)$$

as follows: an object of $\mathcal{D}^b\mathrm{Coh}(X \times Y)$ belongs to $\mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(X \times Y)$ if its cohomology sheaves are supported on a closed subscheme $Z \subset X \times Y$ such that the restrictions to Z of p_X and p_Y are proper. Any $\mathcal{F} \in \mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(X \times Y)$ gives rise to a functor

$$F^{\mathcal{F}}_{X \to Y} : \left\{ \begin{array}{ccc} \mathcal{D}^b\mathrm{Coh}(X) & \to & \mathcal{D}^b\mathrm{Coh}(Y) \\ \mathcal{M} & \mapsto & R(p_Y)_*(\mathcal{F} \overset{L}{\otimes}_{X \times Y} p_X^* \mathcal{M}) \end{array} \right.$$

(use [Har66, II.2.2, II.4.3]). The assignment $\mathcal{F} \mapsto F^{\mathcal{F}}_{X\to Y}$ is functorial.

Let now X, Y and Z be varieties. We define the convolution product

$$* : \mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(Y \times Z) \times \mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(X \times Y) \to \mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(X \times Z)$$

by the formula

$$\mathcal{G} * \mathcal{F} := R(p_{X,Z})_*((p_{X,Y})^*\mathcal{F} \overset{L}{\otimes}_{X\times Y\times Z} (p_{Y,Z})^*\mathcal{G}),$$

where $p_{X,Z}$, $p_{X,Y}$, $p_{Y,Z}$ are the natural projections from $X\times Y\times Z$. The following easy result is classical. It can be proved using flat base change ([Har66, II.5.12]) and the projection formula ([Har66, II.5.6]).

Lemma 2.1.1. *Let $\mathcal{F} \in \mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(X \times Y)$, $\mathcal{G} \in \mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(Y \times Z)$. Then*

$$F^{\mathcal{G}}_{Y\to Z} \circ F^{\mathcal{F}}_{X\to Y} \cong F^{\mathcal{G}*\mathcal{F}}_{X\to Z}.$$

In particular, if $X = Y$, the product $*$ endows $\mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(X \times X)$ with the structure of a monoid, with identity $\Delta_*\mathcal{O}_X$ (where $\Delta : X \to X \times X$ is the diagonal embedding). Moreover, $F^{(-)}_{X\to X}$ is a morphism of monoids from this monoid to the monoid of triangulated functors from $\mathcal{D}^b\mathrm{Coh}(X)$ to itself.

Assume now that X and Y are non-singular varieties (so that every coherent sheaf has a finite resolution by locally free sheaves of finite type, see for instance [Har77, ex. III.6.9]), and let $f : X \to Y$ be a proper morphism. Let $\Gamma_f \subset X \times Y$ be the graph of f (a closed subscheme), and let $\Gamma'_f \subset Y \times X$ be the image of Γ_f under the “swap” morphism $X \times Y \to Y \times X$. Then there exist natural isomorphisms of functors

$$Rf_* \cong F^{\mathcal{O}_{\Gamma_f}}_{X\to Y} \quad \text{and} \quad Lf^* \cong F^{\mathcal{O}_{\Gamma'_f}}_{Y\to X}.$$

Hence we have $Lf^* \circ Rf_* \cong F^{\mathcal{O}_{\Gamma'_f}*\mathcal{O}_{\Gamma_f}}_{X\to X}$, with

$$\mathcal{O}_{\Gamma'_f} * \mathcal{O}_{\Gamma_f} \cong R(p_{X,X})_*(\mathcal{O}_{\Gamma_f\times X} \overset{L}{\otimes}_{X\times Y\times X} \mathcal{O}_{X\times\Gamma'_f}).$$

We also have $\mathrm{Id} \cong F^{\Delta_*\mathcal{O}_X}_{X\to X}$.

We denote by $\delta X \subset X \times Y \times X$ the closed subscheme which is the image of X under $x \mapsto (x, f(x), x)$. The following result follows from classical results in the theory of Fourier-Mukai transforms (see [Căl03, 5.1], [KT07, 4.2]):

Lemma 2.1.2. *The adjunction morphism $Lf^* \circ Rf_* \to \mathrm{Id}$ is induced by the following morphism in $\mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(X \times X)$:*

$$R(p_{X,X})_*(\mathcal{O}_{\Gamma_f\times X} \overset{L}{\otimes}_{X\times Y\times X} \mathcal{O}_{X\times\Gamma'_f}) \to R(p_{X,X})_*(\mathcal{O}_{(\Gamma_f\times X)\cap(X\times\Gamma'_f)})$$
$$\xrightarrow{\mathrm{res}} R(p_{X,X})_*(\mathcal{O}_{\delta X}) \cong \Delta_*\mathcal{O}_X$$

where the second morphism is induced by restriction, and the first one by the natural morphism $\mathcal{O}_{\Gamma_f\times X}\overset{L}{\otimes}_{X\times Y\times X}\mathcal{O}_{X\times\Gamma'_f} \to \mathcal{O}_{\Gamma_f\times X} \otimes_{X\times Y\times X} \mathcal{O}_{X\times\Gamma'_f}$.

We will also need the following Lemma:

Lemma 2.1.3. *Let $\mathcal{F} \in \mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(X \times X)$. Then $\mathcal{O}_{\Gamma_f} * \mathcal{F} \cong R(\mathrm{Id} \times f)_*(\mathcal{F})$.*

Proof. We denote by $p_{i,j}$ the natural projections from $X \times X \times Y$ to $X \times X$ or $X \times Y$, and by $\Delta : X \to X \times X$ the diagonal embedding. Then we have

$$\begin{array}{rcl} \mathcal{O}_{\Gamma_f} * \mathcal{F} & = & R(p_{1,3})_*(p_{1,2}^*\mathcal{F} \overset{L}{\otimes}_{X \times X \times Y} p_{2,3}^*\mathcal{O}_{\Gamma_f}); \\ p_{2,3}^*\mathcal{O}_{\Gamma_f} & \cong & R(\mathrm{Id} \times \mathrm{Id} \times f)_*(\mathrm{Id} \times \Delta)_*\mathcal{O}_{X \times X}. \end{array}$$

Now, by the projection formula, $\mathcal{O}_{\Gamma_f} * \mathcal{F}$ is isomorphic to

$$R(p_{1,3})_* R(\mathrm{Id} \times \mathrm{Id} \times f)_*(\mathrm{Id} \times \Delta)_*(L(\mathrm{Id} \times \Delta)^* L(\mathrm{Id} \times \mathrm{Id} \times f)^*(p_{1,2})^*\mathcal{F}).$$

The result follows, since $(p_{1,3}) \circ (\mathrm{Id} \times \mathrm{Id} \times f) \circ (\mathrm{Id} \times \Delta) = (\mathrm{Id} \times f)$ and $(p_{1,2}) \circ (\mathrm{Id} \times \mathrm{Id} \times f) \circ (\mathrm{Id} \times \Delta) = \mathrm{Id}_{X \times X}$. □

2.2 Action of a group on a category

By an *action* of a group A on a category $\mathcal{C}$ we mean a *weak* action, *i.e.* a group morphism from A to the isomorphism classes of auto-equivalences of the category $\mathcal{C}$ (see [BMR06], [KT07]). We will not consider the problem of the compatibility of the isomorphisms of functors corresponding to products of elements of A. If $\mathcal{C} = \mathcal{D}^b\mathrm{Coh}(X)$ for a variety X, to define such an action it is sufficient to construct a morphism of monoids from A to the monoid of isomorphism classes in $\mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(X \times X)$, endowed with the product $*$.

We will be interested in the case $A = B'_{\mathrm{aff}}$ and $X = \widetilde{\mathfrak{g}}$ or $\widetilde{\mathcal{N}}$. Using the presentation of B'_{aff} that we have given in Theorem 1.1.3, to construct the action we only have to define the kernels corresponding to the generators T_α and θ_x, and to show that they satisfy relations (1) to (4) in $\mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(X \times X)$, up to isomorphism.

2.3 Construction of kernels

Let $\alpha \in \Phi$ be a simple root. In this subsection we construct the kernel for the action of T_α. Here $\mathrm{char}(\Bbbk)$ is arbitrary. First we recall the following well-known formulae for the adjoint action of G on $\mathfrak{g}$, that can be checked in $\mathfrak{sl}(2, \Bbbk)$:

$$\left\{ \begin{array}{c} u_\alpha(x) \cdot e_{-\alpha} = e_{-\alpha} + x h_\alpha - x^2 e_\alpha; \\ n_\alpha u_\alpha(x) \cdot e_{-\alpha} = x^2 e_{-\alpha} - x h_\alpha - e_\alpha; \\ u_\alpha(x) \cdot h_\alpha = h_\alpha - 2x e_\alpha. \end{array} \right. \tag{2.3.1}$$

Let us introduce some notation. If $X \xrightarrow{p} \mathcal{B}$ is a scheme over $\mathcal{B}$ (resp. if $Y \xrightarrow{q} \mathcal{B} \times \mathcal{B}$ is a scheme over $\mathcal{B} \times \mathcal{B}$), and $x, y \in \mathbb{X}$, we denote by $\mathcal{O}_X(x)$ (resp. $\mathcal{O}_Y(x, y)$) the line bundle $p^*\mathcal{O}_{\mathcal{B}}(x)$ (resp. $q^*(\mathcal{O}_{\mathcal{B}}(x) \boxtimes \mathcal{O}_{\mathcal{B}}(y))$). If $\mathcal{F} \in \mathcal{D}^b\mathrm{Coh}(X)$ (resp. $\mathcal{D}^b\mathrm{Coh}(Y)$), we denote by $\mathcal{F}(x)$ (resp. $\mathcal{F}(x, y)$) the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(x)$ (resp. $\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(x, y)$). We use similar notation for schemes over $\mathcal{B} \times \mathcal{B} \times \mathcal{B}$. If $X \xrightarrow{a} Y$ is a scheme over Y, and if $Z \subset Y$

is a locally closed subscheme, we write $X|_Z$ for the inverse image $a^{-1}(Z)$. Similarly, if $\mathcal{F}$ is a sheaf on X, we write $\mathcal{F}|_Z$ for the restriction of $\mathcal{F}$ to $X|_Z$.

Recall the notations of I.1.1. Let us consider the scheme $\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}$. It is reduced, and it can be described as a variety induced from B to G. More precisely, define

$$\mathscr{R}_\alpha := \{(X, gB) \in \mathfrak{g}^* \times (P_\alpha/B) \mid X_{|\mathfrak{n}+g\cdot\mathfrak{n}} = 0\}.$$

We have a natural isomorphism

$$G \times^B \mathscr{R}_\alpha \cong \widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}.$$

To study the variety $\mathscr{R}_\alpha$, we introduce some coordinates. On $\mathfrak{g}^*$ we use the coordinates $\{e_\gamma, \gamma \in R\} \cup \{h_\beta, \beta \in \Phi\}$. Consider the open covering $P_\alpha/B = (U_\alpha B/B) \cup (n_\alpha U_\alpha B/B)$. The morphism u_α induces isomorphisms $\Bbbk \xrightarrow{\sim} U_\alpha \xrightarrow{\sim} U_\alpha B/B$ and $\Bbbk \xrightarrow{\sim} n_\alpha U_\alpha \xrightarrow{\sim} n_\alpha U_\alpha B/B$. We will use the coordinate t on $\Bbbk$. Then $\mathscr{R}_\alpha|_{(U_\alpha B/B)}$ is the set of $(X,t) \in \mathfrak{g}^* \times \Bbbk$ such that X vanishes on e_γ for $\gamma \in R^-$ and on $u_\alpha(t) \cdot e_{-\alpha} = e_{-\alpha} + t h_\alpha - t^2 e_\alpha$ (see (2.3.1)). Similarly, $\mathscr{R}_\alpha|_{(n_\alpha U_\alpha B/B)}$ is the set of $(X,t) \in \mathfrak{g}^* \times \Bbbk$ such that X vanishes on e_γ for $\gamma \in R^-$ and on $n_\alpha u_\alpha(t) \cdot e_{-\alpha} = -e_\alpha - t h_\alpha + t^2 e_{-\alpha}$. These are affine varieties, with respective coordinate rings

$$\Bbbk[\mathscr{R}_\alpha|_{U_\alpha B/B}] \cong \Bbbk[h_\beta, e_\gamma, t, \beta \in \Phi, \gamma \in R^+]/(t(h_\alpha - t e_\alpha))$$
$$\Bbbk[\mathscr{R}_\alpha|_{n_\alpha U_\alpha B/B}] \cong \Bbbk[h_\beta, e_\gamma, t, \beta \in \Phi, \gamma \in R^+]/(e_\alpha + t h_\alpha).$$

In particular, $\mathscr{R}_\alpha$ has two irreducible components: one is

$$\mathscr{D}_\alpha := (\mathfrak{g}/\mathfrak{n})^* \times (B/B) \subset \mathfrak{g}^* \times (P_\alpha/B),$$

and the other one is $\mathscr{S}_\alpha$, the closure of the complement of $\mathscr{D}_\alpha$ in $\mathscr{R}_\alpha$. It is a reduced scheme, and we have the geometric description

$$\mathscr{S}_\alpha = \{(X, gB) \in \mathfrak{g}^* \times (P_\alpha/B) \mid X_{|\mathfrak{n}+g\cdot\mathfrak{n}} = 0 \text{ and } X(h_\alpha) = 0 \text{ if } gB = B\}.$$

Hence $\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}$ has two irreducible components: $\Delta_{\widetilde{\mathfrak{g}}} := G \times^B \mathscr{D}_\alpha$, which is the diagonal embedding of $\widetilde{\mathfrak{g}}$, and $S_\alpha := G \times^B \mathscr{S}_\alpha$. Geometrically,

$$S_\alpha = \left\{ (X, gB, hB) \in \mathfrak{g}^* \times (\mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B}) \;\middle|\; \begin{array}{l} X_{|g\cdot\mathfrak{n}+h\cdot\mathfrak{n}} = 0 \\ \text{and } X(g \cdot h_\alpha) = 0 \text{ if } gB = hB \end{array} \right\}.$$

This second component is a vector bundle over $\mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B}$, of rank $\dim(\mathfrak{g}/\mathfrak{n}) - 1$. In particular, S_α is smooth.

Finally, let us define the closed subscheme S'_α of $\widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}$ by setting

$$S'_\alpha := S_\alpha \cap (\widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}).$$

We will see in section 5 that this intersection is a reduced scheme, hence a variety. S'_α is affine over $\mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B}$, and it is the induced variety of the subvariety $\mathscr{S}'_\alpha$ of $\mathfrak{g}^* \times (P_\alpha/B)$ defined by

$$\mathscr{S}'_\alpha = \{(X, gB) \in \mathfrak{g}^* \times (P_\alpha/B) \mid X_{|\mathfrak{b}+g\cdot\mathfrak{b}} = 0\}.$$

The main result of this chapter is the following:

Theorem 2.3.2. *Assume either that G has no factor of type $\mathbf{G}_2$ and $\mathrm{char}(\Bbbk) \neq 2$ if R is not simply-laced, or that p is very good for G. There exists an action of B'_{aff} on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$ (resp. $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$) for which*

(i) *The action of the element θ_x is given by the convolution with kernel $\Delta_*(\mathcal{O}_{\widetilde{\mathfrak{g}}}(x))$ (resp. $\Delta_*(\mathcal{O}_{\widetilde{\mathcal{N}}}(x))$) for $x \in \mathbb{X}$, where Δ is the diagonal embedding.*

(ii) *The action of the element T_α is given by the convolution with kernel $\mathcal{O}_{S_\alpha}$ (resp. $\mathcal{O}_{S'_\alpha}$) for $\alpha \in \Phi$.*

Moreover, the action of $(T_\alpha)^{-1}$ is the convolution with kernel $\mathcal{O}_{S_\alpha}(-\rho, \rho-\alpha)$ (respectively $\mathcal{O}_{S'_\alpha}(-\rho, \rho-\alpha)$).

These actions correspond under the functor $i_* : \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}) \to \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$, where $i : \widetilde{\mathcal{N}} \hookrightarrow \widetilde{\mathfrak{g}}$ is the closed embedding.

The proof of this result occupies most of the rest of this chapter. It is clear that the kernels $\Delta_*(\mathcal{O}_{\widetilde{\mathfrak{g}}}(x))$ (respectively $\Delta_*(\mathcal{O}_{\widetilde{\mathcal{N}}}(x))$) are invertible, and satisfy relation 2 of Theorem 1.1.3. In subsection 2.4 we show that the kernels $\mathcal{O}_{S_\alpha}$ for $\alpha \in \Phi$ are also invertible, with inverse $\mathcal{O}_{S_\alpha}(-\rho, \rho-\alpha)$. Then, in subsection 2.5 we show that they satisfy relations 3 and 4 of Theorem 1.1.3.

In sections 3 and 4 we show that the kernels satisfy relations 1 of Theorem 1.1.3, under the first assumptions. The relations under the second assumptions are proved later, in section 8. In section 5 we explain how one can deduce the assertions concerning the action on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$.

2.4 Action of the inverse of the generators

In this subsection we fix a simple root $\alpha \in \Phi$. The following lemma is very easy, but useful. This result also appears in [Lus98, 7.19].

Lemma 2.4.1. *Let $\lambda \in \mathbb{X}$, such that $\langle \lambda, \alpha^\vee \rangle = 0$. The line bundle $\mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(\lambda, -\lambda)$ is trivial.*

Proof. We have $\mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(\lambda, -\lambda) \cong \mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(\lambda, 0) \otimes \mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(0, -\lambda)$. Moreover, if $p : \mathcal{B}\times_{\mathcal{P}_\alpha} \mathcal{B} \to \mathcal{P}_\alpha$ denotes the natural morphism, $\mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(\lambda, 0) \cong p^*\mathcal{O}_{\mathcal{P}_\alpha}(\lambda)$ and $\mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(0, -\lambda) \cong p^*\mathcal{O}_{\mathcal{P}_\alpha}(-\lambda)$ The result follows. □

Let us remark in particular that if $\langle \lambda, \alpha^\vee \rangle = \langle \mu, \alpha^\vee \rangle$ then we have $\mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(\lambda, \mu) \cong \mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(\mu, \lambda)$. We deduce that $\mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(-\rho, \rho-\alpha) \cong \mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(\rho-\alpha, -\rho)$, and that $\mathcal{O}_{S_\alpha}(-\rho, \rho-\alpha) \cong \mathcal{O}_{S_\alpha}(\rho-\alpha, -\rho)$, $\mathcal{O}_{S'_\alpha}(-\rho, \rho-\alpha) \cong \mathcal{O}_{S'_\alpha}(\rho-\alpha, -\rho)$.

We will use several times the following result: any finite collection of points of $\mathcal{B}$ is contained in a B-translate of U^+B/B. This follows easily from the fact that if $g_i \in G$ ($i = 1, \dots, n$) then the intersection $(\bigcap_{i=1}^n g_i BU^+) \cap (BU^+)$ is not empty, as an intersection of dense open sets.

Proposition 2.4.2. *There exist isomorphisms in $\mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}})$:*

$$\mathcal{O}_{S_\alpha} * \big(\mathcal{O}_{S_\alpha}(-\rho, \rho-\alpha)\big) \cong \Delta_*\mathcal{O}_{\widetilde{\mathfrak{g}}} \cong \big(\mathcal{O}_{S_\alpha}(-\rho, \rho-\alpha)\big) * \mathcal{O}_{S_\alpha}.$$

Proof. We have $(p_{1,2})^*\mathcal{O}_{S_\alpha} \cong \mathcal{O}_{S_\alpha \times \widetilde{\mathfrak{g}}}$ and $(p_{2,3})^*\mathcal{O}_{S_\alpha} \cong \mathcal{O}_{\widetilde{\mathfrak{g}} \times S_\alpha}$. First, let us show that the tensor product

$$\mathcal{O}_{S_\alpha \times \widetilde{\mathfrak{g}}} \overset{L}{\otimes}_{\widetilde{\mathfrak{g}}^3} \mathcal{O}_{\widetilde{\mathfrak{g}} \times S_\alpha} \tag{2.4.3}$$

is concentrated in degree 0. As each of these varieties over $\mathcal{B}^3$ is the induced variety (from B to G) of its restriction to $(B/B) \times \mathcal{B}^2$, we only have to consider the situation over $(B/B) \times \mathcal{B}^2$. By B-equivariance, we can even restrict to $(B/B) \times (U^+B/B)^2 \cong (U^+)^2$ (see the remark above).

Let us choose coordinates on $\widetilde{\mathfrak{g}}^3|_{(B/B)\times(U^+B/B)^2}$. We have isomorphisms $\widetilde{\mathfrak{g}}|_{U^+B/B} \xrightarrow{\sim} (\mathfrak{b}^+)^* \times (U^+B/B)$ (induced by restriction), and $\widetilde{\mathfrak{g}}|_{B/B} \cong (\mathfrak{b}^+)^*$. Hence on the fibers, isomorphic to $((\mathfrak{b}^+)^*)^3$, we choose coordinates $e_\gamma^{(j)}$, $h_\beta^{(j)}$ ($\gamma \in R^+$, $\beta \in \Phi$, $j \in \{1,2,3\}$) which are copies of the elements of the basis of $\mathfrak{g}$ defined in chapter I. The multiplication induces an isomorphism $U_{(\alpha)}^+ \times U_\alpha \xrightarrow{\sim} U^+$, where $U_{(\alpha)}^+$ is the product of the U_γ for $\gamma \in R^+ - \{\alpha\}$ (this is the unipotent radical of the parabolic subgroup opposite to P_α). Hence, u_α and multiplication induce an isomorphism $U_{(\alpha)}^+ \times \Bbbk \xrightarrow{\sim} U^+$. Using this, we choose coordinates $(u^{(j)}, t^{(j)})$ on U^+, considered as the base of the j-th copy of $\widetilde{\mathfrak{g}}$ ($j = 2,3$).

Then $(S_\alpha \times \widetilde{\mathfrak{g}})|_{(B/B)\times(U^+B/B)^2}$ is defined in $(\widetilde{\mathfrak{g}})^3|_{(B/B)\times(U^+B/B)^2}$ by the equations $u^{(2)} = 1$, $h_\beta^{(1)} = h_\beta^{(2)}$ ($\beta \in \Phi$), $e_\gamma^{(1)} = e_\gamma^{(2)}$ ($\gamma \in R^+$) and $h_\alpha^{(1)} - t^{(2)} e_\alpha^{(1)} = 0$ (see 2.3). It is clear that these equations form a regular sequence in $\Bbbk[\widetilde{\mathfrak{g}}^3|_{(B/B)\times(U^+B/B)^2}]$. Similarly, $(\widetilde{\mathfrak{g}} \times S_\alpha)|_{(B/B)\times(U^+B/B)^2}$ is defined by the equations $u^{(2)} = u^{(3)}$, $h_\beta^{(2)} = h_\beta^{(3)}$ ($\beta \in \Phi$), $e_\gamma^{(2)} = e_\gamma^{(3)}$ ($\gamma \in R^+$) and $u^{(3)} \cdot (h_\alpha^{(3)} - (t^{(2)} + t^{(3)}) e_\alpha^{(3)}) = 0$. Now the union of these two sequences is again a regular sequence, and defines a reduced scheme. Hence the derived tensor product (2.4.3) is concentrated in degree 0, and equals the sheaf of functions on the subvariety $V_\alpha := (S_\alpha \times \widetilde{\mathfrak{g}}) \cap (\widetilde{\mathfrak{g}} \times S_\alpha)$ of $\widetilde{\mathfrak{g}}^3$. Now we compute

$$R(p_{1,3})_*(\mathcal{O}_{V_\alpha}(\rho - \alpha, -\rho, 0)) \quad \text{and} \quad R(p_{1,3})_*(\mathcal{O}_{V_\alpha}(0, -\rho, \rho - \alpha)). \tag{2.4.4}$$

The following result will be proved later:

Lemma 2.4.5. *The variety V_α has two irreducible components: V_α^1, which is the restriction of V_α to the partial diagonal $\Delta_{\mathcal{B}}^{1,3} \subset \mathcal{B}^3$, and V_α^2, which has the following geometric description:*

$$V_\alpha^2 = \{(X, g_1B, g_2B, g_3B) \in \mathfrak{g}^* \times (\mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B}) \mid X_{|g_1 \cdot (\mathfrak{n} + \mathfrak{sl}(2,\alpha))} = 0\}.$$

Moreover, there exist exact sequences of sheaves

$$\mathcal{O}_{V_\alpha^1} \hookrightarrow \mathcal{O}_{V_\alpha}(\rho - \alpha, -\rho, 0) \twoheadrightarrow \mathcal{O}_{V_\alpha^2}(\rho - \alpha, -\rho, 0);$$
$$\mathcal{O}_{V_\alpha^1} \hookrightarrow \mathcal{O}_{V_\alpha}(0, -\rho, \rho - \alpha) \twoheadrightarrow \mathcal{O}_{V_\alpha^2}(0, -\rho, \rho - \alpha).$$

It follows that to compute the direct images (2.4.4) we only have to compute the objects $R(p_{1,3})_*(\mathcal{O}_{V_\alpha^1})$, $R(p_{1,3})_*(\mathcal{O}_{V_\alpha^2}(\rho - \alpha, -\rho, 0))$ and $R(p_{1,3})_*(\mathcal{O}_{V_\alpha^2}(0, -\rho, \rho - \alpha))$. But $R(p_{1,3})_*(\mathcal{O}_{V_\alpha^2}(\rho - \alpha, -\rho, 0)) = R(p_{1,3})_*(\mathcal{O}_{V_\alpha^2}(0, -\rho, \rho - \alpha)) = 0$ because $p_{1,3}$ is a locally trivial fibration of fiber $\mathbb{P}^1_{\Bbbk}$ on V_α^2, and the sheaf on this fiber is $\mathcal{O}_{\mathbb{P}^1}(-1)$. To conclude, we only have to show that $R(p_{1,3})_*(\mathcal{O}_{V_\alpha^1}) \cong \Delta_* \mathcal{O}_{\widetilde{\mathfrak{g}}}$.

By local triviality we only have to consider the morphism

$$q_{1,3} : V_\alpha|_{(B/B)\times(P_\alpha/B)\times(B/B)} \to (\mathfrak{g}/\mathfrak{n})^*.$$

Then define $M := (\mathfrak{sl}(2,\alpha)/(\Bbbk e_{-\alpha}))^*$, and choose a vector subspace $M' \subset \mathfrak{g}/\mathfrak{n}$ such that $\mathfrak{g}/\mathfrak{n} \cong M^* \oplus M'$. Let $E = \{(D,x) \in \mathbb{P}(M) \times M \mid x \in D\}$ be the tautological line bundle on $\mathbb{P}(M)$. Then the morphism $q_{1,3}$ identifies with the product of $\mathrm{Id}_{(M')^*}$ and the canonical projection $f : E \to M$. Hence we only have to show that $Rf_*\mathcal{O}_E \cong \mathcal{O}_M$. As M is affine we only have to consider the global sections; but the direct image of $\mathcal{O}_E$ under the canonical projection to $\mathbb{P}(M)$ is $\bigoplus_{m\geq 0} \mathcal{O}_{\mathbb{P}(M)}(m)$, whose global sections are $S(M^*)$.

This completes the proof of Proposition 2.4.2, assuming Lemma 2.4.5. □

Proof of Lemma 2.4.5. Consider the subvariety $\mathscr{V}_\alpha$ of $\mathfrak{g}^* \times (P_\alpha/B) \times (P_\alpha/B)$:

$$\mathscr{V}_\alpha := \{(X, gB, hB) \in \mathfrak{g}^* \times (P_\alpha/B) \times (P_\alpha/B) \mid X_{|\mathfrak{n}+g\cdot\mathfrak{n}+h\cdot\mathfrak{n}} = 0, \\ X(h_\alpha) = 0 \text{ if } gB = B \text{ and } X(g\cdot h_\alpha) = 0 \text{ if } gB = hB\}.$$

We have an isomorphism $V_\alpha \cong G \times^B \mathscr{V}_\alpha$. On $(P_\alpha/B)^2$ we use the open covering $(P_\alpha/B)^2 = (U_\alpha B/B)^2 \cup (n_\alpha U_\alpha B/B)^2 \cup [(U_\alpha B/B) \times (n_\alpha U_\alpha B/B)] \cup [(n_\alpha U_\alpha B/B) \times (U_\alpha B/B)]$. Each of these open sets is isomorphic to $\Bbbk^2$, via u_α. We use the coordinates $t^{(1)}$ and $t^{(2)}$ on $(P_\alpha/B)^2$, and $\{e_\gamma, \gamma \in R, h_\beta, \beta \in \Phi\}$ on $\mathfrak{g}^*$. The change of coordinates on the intersection $(U_\alpha B/B) \cap (n_\alpha U_\alpha B/B)$ is given by $t \mapsto -\frac{1}{t}$ (this can be checked in $\mathrm{SL}(2,\Bbbk)$).

The restriction $\mathscr{V}_\alpha|_{(U_\alpha B/B)^2}$ is defined in $\mathfrak{g}^* \times \Bbbk^2$ by the equations $e_\gamma = 0$ ($\gamma \in R^-$), $h_\alpha - t^{(1)} e_\alpha = 0$ and $h_\alpha - (t^{(1)} + t^{(2)}) e_\alpha = 0$ (see the preceding proof). This last equation can be replaced by $t^{(2)} e_\alpha = 0$. Similarly, $\mathscr{V}_\alpha|_{(n_\alpha U_\alpha B/B)^2}$ is defined in $\mathfrak{g}^* \times \Bbbk^2$ by the equations $e_\gamma = 0$ ($\gamma \in R^-$), $e_\alpha + t^{(1)} h_\alpha = 0$ and $h_\alpha = 0$. Over $(U_\alpha B/B) \times (n_\alpha U_\alpha B/B)$, the equations are $e_\gamma = 0$ ($\gamma \in R^-$), $h_\alpha - t^{(1)} e_\alpha = 0$ and $e_\alpha = 0$. Finally, $\mathscr{V}_\alpha|_{(n_\alpha U_\alpha B/B)\times(U_\alpha B/B)}$ is defined by $e_\gamma = 0$ ($\gamma \in R^-$), $e_\alpha + t^{(1)} h_\alpha = 0$ and $t^{(2)} h_\alpha = 0$. These equations show that $\mathscr{V}_\alpha$ has two irreducible components: $\mathscr{V}_\alpha^1$, which is the restriction of $\mathscr{V}_\alpha$ to $(P_\alpha/B) \times (B/B) \subset (P_\alpha/B)^2$, and $\mathscr{V}_\alpha^2$, which has the following geometric description:

$$\mathscr{V}_\alpha^2 = \{(X, gB, hB) \in \mathfrak{g}^* \times (P_\alpha/B)^2 \mid X_{|\mathfrak{n}+\mathfrak{sl}(2,\alpha)} = 0\}.$$

The varieties $\mathscr{V}_\alpha$, $\mathscr{V}_\alpha^1$ and $\mathscr{V}_\alpha^2$ are affine over $(U_\alpha B/B)^2$, with respective rings of functions $\Bbbk[e_\gamma, h_\beta, t^{(i)}]/(h_\alpha - t^{(1)} e_\alpha,\ t^{(2)} e_\alpha)$, $\Bbbk[e_\gamma, h_\beta, t^{(i)}]/(h_\alpha - t^{(1)} e_\alpha,\ t^{(2)})$, $\Bbbk[e_\gamma, h_\beta, t^{(i)}]/(h_\alpha - t^{(1)} e_\alpha,\ e_\alpha)$. Hence the multiplication by e_α and the natural quotient induce an exact sequence of sheaves

$$\mathcal{O}_{\mathscr{V}_\alpha^1}|_{(U_\alpha B/B)^2} \hookrightarrow \mathcal{O}_{\mathscr{V}_\alpha}|_{(U_\alpha B/B)^2} \twoheadrightarrow \mathcal{O}_{\mathscr{V}_\alpha^2}|_{(U_\alpha B/B)^2}.$$

Multiplication by h_α induces a similar sequence on $(n_\alpha U_\alpha B/B) \times (U_\alpha B/B)$.

The element $e_\alpha \in \Bbbk[\mathscr{V}_\alpha|_{(U_\alpha B/B)^2}]$ goes to 0 when restricted to the open sets $(n_\alpha U_\alpha B/B)^2$ or $(U_\alpha B/B) \times (n_\alpha U_\alpha B/B)$, and to $-t^{(1)} h_\alpha$ when restricted to $(n_\alpha U_\alpha B/B) \times (U_\alpha B/B)$.

Hence the preceding exact sequences glue to give an exact sequence of (non B-equivariant) sheaves

$$\mathcal{O}_{\mathscr{V}_\alpha^1} \otimes_{\mathcal{O}_{(P_\alpha/B)^2}} \mathcal{O}_{(P_\alpha/B)^2}(1,0) \hookrightarrow \mathcal{O}_{\mathscr{V}_\alpha} \twoheadrightarrow \mathcal{O}_{\mathscr{V}_\alpha^2}$$

where we have used the isomorphism $P_\alpha/B \cong \mathbb{P}^1_{\Bbbk}$. Now consider the B-equivariant structures. The second morphism in this sequence is obviously equivariant. We have $\mathcal{O}_{(P_\alpha/B)^2}(1,0) = \mathcal{O}_{(P_\alpha/B)^2}(\rho, 0)$, and the first arrow of the exact sequence comes by definition from a B-equivariant morphism $\Bbbk_B(\alpha-\rho) \otimes_{\Bbbk} \mathcal{O}_{\mathscr{V}_\alpha^1}(\rho,0) \hookrightarrow \mathcal{O}_{\mathscr{V}_\alpha}$. Hence we obtain the exact sequence of B-equivariant sheaves

$$\Bbbk_B(\alpha-\rho) \otimes_{\Bbbk} \mathcal{O}_{\mathscr{V}_\alpha^1}(\rho,0) \hookrightarrow \mathcal{O}_{\mathscr{V}_\alpha} \twoheadrightarrow \mathcal{O}_{\mathscr{V}_\alpha^2}.$$

Inducing from B to G, this gives the first exact sequence of the lemma. To prove the second one, we observe that we also have an exact sequence

$$\mathcal{O}_{V_\alpha^1}(\alpha-\rho,0,\rho-\alpha) \hookrightarrow \mathcal{O}_{V_\alpha}(0,-\rho,\rho-\alpha) \twoheadrightarrow \mathcal{O}_{V_\alpha^2}(0,-\rho,\rho-\alpha).$$

As V_α^1 is supported on $\Delta_{\mathcal{B}}^{1,3} \subset \mathcal{B}^3$, the first sheaf equals $\mathcal{O}_{V_\alpha^1}$. □

Remark 2.4.6. In these two results, one can replace ρ by any $\lambda \in \mathbb{X}$ such that $\langle \lambda, \alpha^\vee \rangle = 1$. This follows either from the proofs, or from Lemma 2.4.1.

2.5 First relations

In this subsection we show that the kernels of Theorem 2.3.2 for the action on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$ satisfy relations 3 and 4 of the presentation of B'_{aff} given by Theorem 1.1.3.

Let us consider relation 3. Let $\alpha \in \Phi$ and $x \in \mathbb{X}$ be such that $\langle x, \alpha^\vee \rangle = 0$. We have to show that $\mathcal{O}_{S_\alpha}$ commutes with $\Delta_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(x)$. But

$$\mathcal{O}_{S_\alpha} * (\Delta_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(x)) \cong \mathcal{O}_{S_\alpha}(x,0), \quad (\Delta_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(x)) * \mathcal{O}_{S_\alpha} \cong \mathcal{O}_{S_\alpha}(0,x),$$

and $\mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(x,0) = \mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(0,x)$ by Lemma 2.4.1. Taking the inverse image to S_α, we obtain the result.

Now, consider relation 4. Let $\alpha \in \Phi$ and $x \in \mathbb{X}$ be such that $\langle x, \alpha^\vee \rangle = 1$. We have to prove that $\Delta_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(x) \cong \mathcal{O}_{S_\alpha} * (\Delta_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(x-\alpha)) * \mathcal{O}_{S_\alpha}$. Due to Proposition 2.4.2, this is equivalent to proving

$$(\Delta_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(x)) * (\mathcal{O}_{S_\alpha}(-\rho,\rho-\alpha)) \cong (\mathcal{O}_{S_\alpha}) * (\Delta_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(x-\alpha)).$$

We have $\mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(-\rho, x+\rho-\alpha) \cong \mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(x-\alpha,0)$ by Lemma 2.4.1 again. The result follows, since

$$\begin{aligned}(\Delta_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(x)) * (\mathcal{O}_{S_\alpha}(-\rho,\rho-\alpha)) &\cong \mathcal{O}_{S_\alpha}(-\rho,x+\rho-\alpha),\\ (\mathcal{O}_{S_\alpha}) * (\Delta_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(x-\alpha)) &\cong \mathcal{O}_{S_\alpha}(x-\alpha,0).\end{aligned}$$

2.6 More notation

In this subsection we introduce notation concerning Schubert varieties and Demazure resolutions (following [BK04]).

If $w \in W$, we denote by X_w the corresponding Schubert variety. This is the closure of BwB/B in $\mathcal{B}$. Similarly, we denote by $\mathcal{X}_w$ the closure of the G-orbit of $(B/B, wB/B)$ in $\mathcal{B} \times \mathcal{B}$, called G-Schubert variety. Its points are the pairs of Borel subgroups of G in relative position at most w in the Bruhat order. It identifies with $G \times^B X_w$ under the isomorphism $G \times^B \mathcal{B} \cong \mathcal{B} \times \mathcal{B}$.

For $w = s_1 \cdots s_n$ a reduced expression in W, let $Z_{(s_1, \cdots, s_n)}$ be the associated Demazure resolution of the Schubert variety X_w (as defined in [BK04]). Let also $\mathcal{Z}_{(s_1, \cdots, s_n)}$ be the induction from B to G of this resolution, which is a resolution of $\mathcal{X}_w$, and let $\Phi_{(s_1, \cdots, s_n)} : \mathcal{Z}_{(s_1, \cdots, s_n)} \to \mathcal{X}_w$ be the associated morphism. If s_j is the reflection associated with the simple root $\alpha_j \in \Phi$ for any $j = 1, \ldots, n$, and $\mathcal{P}_j := G/P_j$ for P_j the standard parabolic subgroup of G of type $\{\alpha_j\}$, then we have an isomorphism $\mathcal{Z}_{(s_1, \cdots, s_n)} \cong \mathcal{B} \times_{\mathcal{P}_1} \mathcal{B} \times_{\mathcal{P}_2} \cdots \times_{\mathcal{P}_n} \mathcal{B}$, and $\Phi_{(s_1, \ldots, s_n)}$ identifies with the restriction of the projection $p_{1,n+1} : \mathcal{B}^{n+1} \to \mathcal{B}^2$. Let $\widetilde{\mathcal{Z}}_{(s_1, \cdots, s_n)}$ be the intersection

$$(S_{\alpha_1} \times \widetilde{\mathfrak{g}}^{n-1}) \cap (\widetilde{\mathfrak{g}} \times S_{\alpha_2} \times \widetilde{\mathfrak{g}}^{n-2}) \cap \cdots \cap (\widetilde{\mathfrak{g}}^{n-1} \times S_{\alpha_n}),$$

a closed subscheme of $\widetilde{\mathfrak{g}}^{n+1}$. It is not clear to us what the properties of this scheme are in general. We will show in the cases of interest to us that it is reduced and irreducible.

In the next two sections we prove the finite braid relations, first in the case when the simple roots α and β generate a root system of type $\mathbf{A}_2$, and then in the case when they generate a system of type $\mathbf{B}_2$. The much easier case of a root system of type $\mathbf{A}_1 \times \mathbf{A}_1$ is left to the reader.

3 Finite braid relations for type $\mathbf{A}_2$

Let α and β be simple roots generating a root system of type $\mathbf{A}_2$, *i.e.* such that $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -1$. It is well-known (see *e.g.* [Spr98, 8.2.3]) that there exists $c \in \Bbbk^\times$ such that

$$\forall x, y \in \Bbbk, \quad (u_\alpha(x), u_\beta(y)) = u_{\alpha+\beta}(cxy).$$

Here, "$(-,-)$" is the commutator in G. The following formulae for the adjoint action of G on $\mathfrak{g}$ follow easily:

$$\begin{aligned} u_\alpha(x) \cdot e_\beta &= e_\beta + cxe_{\alpha+\beta}, & u_{\alpha+\beta}(x) \cdot h_\beta &= h_\beta - xe_{\alpha+\beta}, \\ u_\alpha(x) \cdot h_\beta &= h_\beta + xe_\alpha, & u_{\alpha+\beta}(x) \cdot e_{-\beta} &= e_{-\beta} + (x/c)e_\alpha. \end{aligned}$$

We also have $[e_\alpha, e_\beta] = ce_{\alpha+\beta}$. The corresponding formulae with α and β interchanged are obtained by replacing c by $-c$. Note finally that $h_{\alpha+\beta} = h_{s_\alpha(\beta)} = s_\alpha(h_\beta) = h_\alpha + h_\beta$.

In this section we prove that

$$\mathcal{O}_{S_\alpha} * \mathcal{O}_{S_\beta} * \mathcal{O}_{S_\alpha} = R(p_{1,4})_*(\mathcal{O}_{S_\alpha \times \widetilde{\mathfrak{g}}^2} \overset{L}{\otimes}_{\widetilde{\mathfrak{g}}^4} \mathcal{O}_{\widetilde{\mathfrak{g}} \times S_\beta \times \widetilde{\mathfrak{g}}} \overset{L}{\otimes}_{\widetilde{\mathfrak{g}}^4} \mathcal{O}_{\widetilde{\mathfrak{g}}^2 \times S_\alpha})$$

is invariant under the exchange of α and β (where $p_{1,4}:\widetilde{\mathfrak{g}}^4\to\widetilde{\mathfrak{g}}^2$ is the natural projection). In fact we compute this complex of sheaves explicitly.

3.1 Derived tensor product

Lemma 3.1.1. *There exist isomorphisms*

$$\mathcal{O}_{S_\alpha\times\widetilde{\mathfrak{g}}^2}\overset{L}{\otimes}_{\widetilde{\mathfrak{g}}^4}\mathcal{O}_{\widetilde{\mathfrak{g}}\times S_\beta\times\widetilde{\mathfrak{g}}}\overset{L}{\otimes}_{\widetilde{\mathfrak{g}}^4}\mathcal{O}_{\widetilde{\mathfrak{g}}^2\times S_\alpha}\cong\mathcal{O}_{\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}};$$
$$\mathcal{O}_{S_\beta\times\widetilde{\mathfrak{g}}^2}\overset{L}{\otimes}_{\widetilde{\mathfrak{g}}^4}\mathcal{O}_{\widetilde{\mathfrak{g}}\times S_\alpha\times\widetilde{\mathfrak{g}}}\overset{L}{\otimes}_{\widetilde{\mathfrak{g}}^4}\mathcal{O}_{\widetilde{\mathfrak{g}}^2\times S_\beta}\cong\mathcal{O}_{\widetilde{\mathcal{Z}}_{(s_\beta,s_\alpha,s_\beta)}}.$$

Moreover, the schemes $\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}$ and $\widetilde{\mathcal{Z}}_{(s_\beta,s_\alpha,s_\beta)}$ are integral, i.e. *reduced and irreducible.*

Proof. We write the proof in the first case only, the second one being similar (replace c by $-c$). As in the proof of Proposition 2.4.2, we only have to study the situation over $(B/B)\times(U^+B/B)^3$. Let us choose an order on R^+ such that the last three roots are $\alpha+\beta,\beta,\alpha$ (in this order). Let P_α, P_β, $P_{\alpha,\beta}$ be the standard parabolic subgroups of G associated to $\{\alpha\}$, $\{\beta\}$ and $\{\alpha,\beta\}$. We denote by $U^+_{(\alpha)}$, $U^+_{(\beta)}$, $U^+_{(\alpha,\beta)}$ the product of the U_γ for $\gamma\in R^+-\{\alpha\}$, $\gamma\in R^+-\{\beta\}$, $\gamma\in R^+-\{\alpha,\beta,\alpha+\beta\}$ (these are the unipotent radicals of the parabolic subgroups opposite to P_α, P_β, $P_{\alpha,\beta}$). We have an isomorphism $U^+\cong\prod_{\gamma\in R^+}U_\gamma$. Via this isomorphism, the restriction to U^+B/B of the projections $G/B\to G/P_\alpha$ and $G/B\to G/P_{\alpha,\beta}$ become the natural projections $U^+_{(\alpha)}\times U_\alpha\to U^+_{(\alpha)}$ and $U^+_{(\alpha,\beta)}\times U_{\alpha+\beta}\times U_\beta\times U_\alpha\to U^+_{(\alpha,\beta)}$. The restriction of the projection $G/B\to G/P_\beta$ becomes

$$\left\{\begin{array}{ccc} U^+_{(\alpha,\beta)}\times U_{\alpha+\beta}\times U_\beta\times U_\alpha & \to & U^+_{(\beta)}\cong U^+_{(\alpha,\beta)}\times U_{\alpha+\beta}\times U_\alpha \\ (u,u_{\alpha+\beta}(x),u_\beta(y),u_\alpha(z)) & \mapsto & (u,u_{\alpha+\beta}(x-cyz),u_\alpha(z)) \end{array}\right. .$$

As in Proposition 2.4.2, as coordinates on $\widetilde{\mathfrak{g}}^4|_{(B/B)\times(U^+B/B)^3}$ we use $u^{(j)}\in U^+_{(\alpha,\beta)}$, $x^{(j)}$, $y^{(j)}$, $z^{(j)}\in\Bbbk$ on the base, and $h^{(j)}_\delta$ ($\delta\in\Phi$) and $e^{(j)}_\gamma$ ($\gamma\in R^+$) on the fiber of the j-th copy of $\widetilde{\mathfrak{g}}$ (we do not use the coordinates $u^{(1)}$, $x^{(1)}$, $y^{(1)}$ and $z^{(1)}$ because in the first copy of $\widetilde{\mathfrak{g}}$ we only consider the fiber over B/B).

In these coordinates, $(S_\alpha\times\widetilde{\mathfrak{g}}^2)|_{(B/B)\times(U^+B/B)^3}\subset(\widetilde{\mathfrak{g}}^4)|_{(B/B)\times(U^+B/B)^3}$ is defined by the equations

$$(*)\qquad u^{(2)}=1,\ x^{(2)}=0,\ y^{(2)}=0,\ h^{(1)}_\delta=h^{(2)}_\delta,\ e^{(1)}_\gamma=e^{(2)}_\gamma\ (\delta\in\Phi,\gamma\in R^+)$$

$$\text{and}\quad h^{(1)}_\alpha-z^{(2)}e^{(1)}_\alpha=0. \tag{3.1.2}$$

Similarly, $(\widetilde{\mathfrak{g}}\times S_\beta\times\widetilde{\mathfrak{g}})|_{(B/B)\times(U^+B/B)^3}\subset(\widetilde{\mathfrak{g}}^4)|_{(B/B)\times(U^+B/B)^3}$ is defined by the equations

$$(*')\qquad \left\{\begin{array}{l} u^{(3)}=u^{(2)},\ x^{(2)}-cy^{(2)}z^{(2)}=x^{(3)}-cy^{(3)}z^{(3)}, \\ z^{(2)}=z^{(3)},\ h^{(2)}_\delta=h^{(3)}_\delta,\ e^{(2)}_\gamma=e^{(3)}_\gamma \end{array}\right.$$

and $u^{(2)} \cdot u_{\alpha+\beta}(x^{(2)} - cy^{(2)}z^{(2)}) \cdot u_\alpha(z^{(2)}) \cdot (h_\beta^{(2)} - (y^{(2)} + y^{(3)})e_\beta^{(2)}) = 0$, *i.e.*

$$u^{(2)} \cdot \big(h_\beta^{(2)} + z^{(2)}e_\alpha^{(2)} - (y^{(2)} + y^{(3)})e_\beta^{(2)} - (x^{(2)} + cy^{(3)}z^{(2)})e_{\alpha+\beta}^{(2)}\big) = 0. \tag{3.1.3}$$

And finally $(\widetilde{\mathfrak{g}}^2 \times S_\alpha)|_{(B/B)\times(U^+B/B)^3} \subset (\widetilde{\mathfrak{g}}^4)|_{(B/B)\times(U^+B/B)^3}$ is defined by

$$(*'') \qquad u^{(3)} = u^{(4)},\ x^{(3)} = x^{(4)},\ y^{(3)} = y^{(4)},\ h_\delta^{(3)} = h_\delta^{(4)},\ e_\gamma^{(3)} = e_\gamma^{(4)}$$

and $u^{(3)} \cdot u_{\alpha+\beta}(x^{(3)}) \cdot u_\beta(y^{(3)}) \cdot (h_\alpha^{(3)} - (z^{(3)} + z^{(4)})e_\alpha^{(3)}) = 0$, *i.e.*

$$u^{(3)} \cdot \big(h_\alpha^{(3)} - (z^{(3)} + z^{(4)})e_\alpha^{(3)} + y^{(3)}e_\beta^{(3)} + (-x^{(3)} + cy^{(3)}(z^{(3)} + z^{(4)}))e_{\alpha+\beta}^{(3)}\big) = 0. \tag{3.1.4}$$

In each case, the given equations form a regular sequence in $\Bbbk[\widetilde{\mathfrak{g}}^4|_{(B/B)\times(U^+B/B)^3}]$. Let us prove that the union of these equations again forms a regular sequence. First, equations $(*)$, $(*')$ and $(*'')$ allow us to identify all the coordinates in the fibers (we will thus remove the superscript on them), and to eliminate the coordinates $u^{(j)}$, $x^{(2)}$, $y^{(2)}$, $x^{(3)}$, $z^{(2)}$, $x^{(4)}$, $y^{(3)}$. Then equations (3.1.2), (3.1.3) allow to eliminate h_α and h_β, while (3.1.4) becomes $-z^{(4)}e_\alpha + y^{(4)}e_\beta + cy^{(4)}z^{(4)}e_{\alpha+\beta} = 0$, a non-zero equation in the remaining variables. Hence the equations indeed form a regular sequence, and thus the derived tensor product is concentrated in degree 0.

Moreover, the polynomial $-z^{(4)}e_\alpha + y^{(4)}e_\beta + cy^{(4)}z^{(4)}e_{\alpha+\beta}$ is irreducible (it is of degree 1 in e_α, and not divisible by $z^{(4)}$). Hence it defines an integral scheme. Thus the restriction of $(S_\alpha \times \widetilde{\mathfrak{g}}^2) \cap (\widetilde{\mathfrak{g}} \times S_\beta \times \widetilde{\mathfrak{g}}) \cap (\widetilde{\mathfrak{g}}^2 \times S_\alpha)$ to $(B/B) \times (U^+B/B)$ is integral. It follows that the restriction of this scheme to any B-translate of $(B/B) \times (U^+B/B)$ is also integral. Hence $(S_\alpha \times \widetilde{\mathfrak{g}}^2) \cap (\widetilde{\mathfrak{g}} \times S_\beta \times \widetilde{\mathfrak{g}}) \cap (\widetilde{\mathfrak{g}}^2 \times S_\alpha)$ is the union of some integral open sets, each one intersecting each other one. Hence it is integral. □

3.2 Determination of the image

Now we have to show that

$$R(p_{1,4})_*(\mathcal{O}_{(S_\alpha\times\widetilde{\mathfrak{g}}^2)\cap(\widetilde{\mathfrak{g}}\times S_\beta\times\widetilde{\mathfrak{g}})\cap(\widetilde{\mathfrak{g}}^2\times S_\alpha)}) = R(p_{1,4})_*(\mathcal{O}_{\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}})$$

is invariant under the exchange of α and β. First, as the intersection we consider is reduced, we can work with varieties instead of schemes. In this subsection we compute the image of $\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}$ under $p_{1,4}$, and observe that it is invariant under the exchange of α and β (though the variety $\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}$ is of course not). Then we show (in 3.4) that $R(p_{1,4})_*(\mathcal{O}_{\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}})$ is the sheaf of functions on this image.

So, let us consider $p_{1,4}(\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)})$. It is a closed subvariety of $\widetilde{\mathfrak{g}}^2$. Indeed, we have the following diagram, where all the injections are closed immersions:

$$\begin{tikzcd}
\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)} \arrow[r, hook, "i"] & \mathfrak{g}^* \times \mathcal{B}^4 \arrow[r, hook, "j"] \arrow[d, "\pi"] & (\mathfrak{g}^*)^4 \times \mathcal{B}^4 \arrow[d] & \widetilde{\mathfrak{g}}^4 \arrow[l, hook', "\sigma"'] \arrow[d, "p_{1,4}"] \\
 & \mathfrak{g}^* \times \mathcal{B}^2 \arrow[r, hook, "\tau"] & (\mathfrak{g}^*)^2 \times \mathcal{B}^2 & \widetilde{\mathfrak{g}}^2. \arrow[l, hook', "\xi"']
\end{tikzcd}$$

One has $ji(\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}) \subseteq \sigma(\widetilde{\mathfrak{g}}^4)$, and $\xi p_{1,4}(\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}) = \tau\pi i(\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)})$. The morphism π being proper, hence closed, the result follows.

Now we compute explicitly $p_{1,4}(\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)})$ as a subset of $\mathfrak{g}^* \times \mathcal{B}^2$, using the geometric description of S_α and S_β (see 2.3). By G-equivariance, we only have to calculate this image over the points $(B/B, wB/B)$ for w in the subgroup of W generated by s_α and s_β. Recall that the Demazure resolution $\Phi_{(s_\alpha,s_\beta,s_\alpha)}$ is an isomorphism over the complement of $\mathcal{X}_{s_\alpha}$. Hence if $w = s_\alpha^{a_1} s_\beta s_\alpha^{a_2}$ with $a_i \in \{0,1\}$ then for $X \in \mathfrak{g}^*$ the point $(X, B/B, wB/B)$ is in the image if and only if $(X, B/B, s_\alpha^{a_1}B/B, s_\alpha^{a_1}s_\beta B/B, s_\alpha^{a_1}s_\beta s_\alpha^{a_2}B/B)$ is in $\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}$. Using the geometric description of S_α, one obtains the condition on X in cases (i) to (iv):

(i) Fiber over $(B/B, s_\alpha s_\beta s_\alpha B/B)$: $X_{|\mathfrak{n}\oplus\Bbbk e_\alpha\oplus\Bbbk e_\beta\oplus\Bbbk e_{\alpha+\beta}} = 0$.

(ii) Fiber over $(B/B, s_\beta s_\alpha B/B)$: $X_{|\mathfrak{n}\oplus\Bbbk h_\alpha\oplus\Bbbk e_\beta\oplus\Bbbk e_{\alpha+\beta}} = 0$.

(iii) Fiber over $(B/B, s_\alpha s_\beta B/B)$: $X_{|\mathfrak{n}\oplus\Bbbk h_\beta\oplus\Bbbk e_\alpha\oplus\Bbbk e_{\alpha+\beta}} = 0$ (observe that $s_\alpha s_\beta \cdot h_\alpha = h_{s_\alpha s_\beta(\alpha)} = h_\beta$).

(iv) Fiber over $(B/B, s_\beta B/B)$: $X_{|\mathfrak{n}\oplus\Bbbk h_\alpha\oplus\Bbbk h_\beta\oplus\Bbbk e_\beta} = 0$.

(v) Fiber over $(B/B, s_\alpha B/B)$: here the fiber of $\Phi_{(s_\alpha,s_\beta,s_\alpha)}$ is isomorphic to $\mathbb{P}^1_\Bbbk$, with points the $(B/B, gB/B, gB/B, s_\alpha B/B)$ for $g \in P_\alpha$. First, if $g \in s_\alpha B$, the condition on X for $(X, B/B, gB/B, gB/B, s_\alpha B/B)$ to be in the intersection is $X_{|\mathfrak{n}\oplus\Bbbk e_\alpha\oplus\Bbbk(s_\alpha\cdot h_\alpha)\oplus\Bbbk(s_\alpha\cdot h_\beta)} = 0$, *i.e.* $X_{|\mathfrak{n}\oplus\Bbbk e_\alpha\oplus\Bbbk h_\alpha\oplus\Bbbk h_\beta} = 0$. Then, if $g \notin s_\alpha B$, we can assume $g = u_\alpha(\epsilon)$ for some $\epsilon \in \Bbbk$. Then the corresponding condition on X is to vanish on $\mathfrak{n}$ and on

$$h_\alpha - \epsilon e_\alpha, \quad u_\alpha(\epsilon)\cdot h_\beta = h_\beta + \epsilon e_\alpha \quad \text{and} \quad e_\alpha.$$

Hence the condition is the same in the two cases. And finally the condition on X for $(X, B/B, s_\beta B/B)$ to be in $p_{1,4}(\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)})$ is

$$X_{|\mathfrak{n}\oplus\Bbbk h_\alpha\oplus\Bbbk h_\beta\oplus\Bbbk e_\alpha} = 0.$$

(vi) Fiber over $(B/B, B/B)$: the fiber of $\Phi_{(s_\alpha,s_\beta,s_\alpha)}$ over $(B/B, B/B)$ is again $\mathbb{P}^1_\Bbbk$, with points the $(B/B, gB/B, gB/B, B/B)$ for $g \in P_\alpha$. Firstly, if $g \in s_\alpha B/B$, the corresponding condition on X is $X_{|\mathfrak{n}\oplus\Bbbk e_\alpha\oplus\Bbbk h_{\alpha+\beta}} = 0$. Secondly, if $g \notin s_\alpha B$, then we can assume $g = u_\alpha(\epsilon)$ for some $\epsilon \in \Bbbk$. The condition on X is then to vanish on $\mathfrak{n}$, on $h_\alpha - \epsilon e_\alpha$ and on $u_\alpha(\epsilon)\cdot h_\beta = h_\beta + \epsilon e_\alpha$. This is equivalent to vanishing on $\mathfrak{n}$, $h_\alpha - \epsilon e_\alpha$ and $h_\alpha + h_\beta = h_{\alpha+\beta}$. Finally, the condition on X for the point $(X, B/B, B/B)$ to be in the image of $\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}$ under $p_{1,4}$ is that $X_{|\mathfrak{n}\oplus\Bbbk h_{\alpha+\beta}} = 0$, and that either $X(e_\alpha) = 0$, or $X(h_\alpha - \epsilon e_\alpha) = 0$ for some $\epsilon \in \Bbbk$. But if $X(e_\alpha) \neq 0$ then $X(h_\alpha - \epsilon e_\alpha) = 0$ for $\epsilon = X(h_\alpha)/X(e_\alpha)$. So the condition on X is only

$$X_{|\mathfrak{n}\oplus\Bbbk h_{\alpha+\beta}} = 0.$$

These considerations show that $p_{1,4}(\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)})$ is a closed subvariety of $\mathfrak{g}^* \times \mathcal{B} \times \mathcal{B}$, invariant under the exchange of α, β (the computations with α and β interchanged are the same, replacing c by $-c$). We denote it by $S_{\{\alpha,\beta\}}$.

3.3 Normality of $S_{\{\alpha,\beta\}}$

Proposition 3.3.1. *The variety $S_{\{\alpha,\beta\}}$ is integral, normal and Cohen-Macaulay.*

Proof.[6] First, $S_{\{\alpha,\beta\}}$ is integral because it is the image of $\widetilde{\widetilde{\mathcal{Z}}}_{(s_\alpha,s_\beta,s_\alpha)}$, which is integral by Lemma 3.1.1.

For the other properties, as usual, we only have to consider the situation over $(B/B)\times(U^+B/B)$. We keep the notation of the proof of Lemma 3.1.1, and define $\gamma := \alpha+\beta$. Recall the isomorphism $U^+ \cong U^+_{(\alpha,\beta)} \times U_\gamma \times U_\beta \times U_\alpha$ (see the proof of Lemma 3.1.1). As $S_{\{\alpha,\beta\}}$ is supported over $\mathcal{B}\times_{G/P_{\alpha,\beta}}\mathcal{B}$, in fact we only have to consider the situation over $(B/B)\times(U_\gamma U_\beta U_\alpha B/B)\cong U_\gamma\times U_\beta\times U_\alpha$.

Consider a point

$$u = u_\gamma(x_\gamma)u_\beta(x_\beta)u_\alpha(x_\alpha)\in U_\gamma U_\beta U_\alpha,$$

with $x_\gamma x_\beta x_\alpha \neq 0$ and $x_\gamma - cx_\alpha x_\beta \neq 0$. It can also by written

$$u_\alpha(x)u_\beta(y)u_\alpha(z)$$

with $x_\gamma = cxy$, $x_\beta = y$, $x_\alpha = x+z$ (here $xyz\neq 0$). If $X\in(\mathfrak{g}/\mathfrak{n})^*$, and $(X, B/B, uB/B)$ is in $S_{\{\alpha,\beta\}}$, then X must vanish on $u_\alpha(x)\cdot e_{-\alpha} = e_{-\alpha}+xh_\alpha - x^2e_\alpha$, hence on

$$h_\alpha - xe_\alpha. \tag{3.3.2}$$

It must also vanish on $u_\alpha(x)u_\beta(y)\cdot e_{-\beta}$, hence on

$$h_\beta + xe_\alpha - ye_\beta - cxye_\gamma. \tag{3.3.3}$$

Finally, it must vanish on $u_\alpha(x)u_\beta(y)u_\alpha(z)\cdot e_{-\alpha}$, hence on

$$(x+z)h_\alpha - (x+z)^2e_\alpha + yze_\beta + cyz(x+z)e_{\alpha+\beta}.$$

Substracting $(x+z)$ times equation (3.3.2), and dividing by z, we obtain that X must vanish on

$$(x+z)e_\alpha - ye_\beta - cy(x+z)e_\gamma. \tag{3.3.4}$$

The sum of equations (3.3.2) and (3.3.3) becomes

$$h_\alpha + h_\beta - x_\beta e_\beta - x_\gamma e_\gamma. \tag{3.3.5}$$

Multiplying (3.3.2) by $cx_\beta = cy$ gives

$$cx_\beta h_\alpha - x_\gamma e_\alpha. \tag{3.3.6}$$

Equation (3.3.4) can be rewritten

$$x_\alpha e_\alpha - x_\beta e_\beta - cx_\alpha x_\beta e_\gamma. \tag{3.3.7}$$

Finally, adding cx_α times (3.3.2) and cx times (3.3.4) gives

$$cx_\alpha h_\alpha - x_\gamma e_\beta - cx_\alpha x_\gamma e_\gamma. \tag{3.3.8}$$

[6] This proof is due to Patrick Polo.

Let us denote by M the closed subscheme of $\mathbb{A}^{\dim(\mathfrak{g}/\mathfrak{n})+3}$ defined by equations (3.3.5) to (3.3.8). Equation (3.3.5) allows to eliminate h_β; that is, setting $e = x_\alpha$, $f = x_\beta$, $g = x_\gamma$, $h = ch_\alpha$, $i = e_\alpha$, $j = e_\beta$ and $k = ce_\gamma$, we obtain that the coordinate ring of M is a polynomial ring over $A := \Bbbk[e, f, g, h, i, j, k]/(F, G, H)$, where

$$\begin{cases} F &= fh - gi, \\ G &= ei - fj - efk, \\ H &= eh - gj - egk. \end{cases}$$

Lemma 3.3.9. *A is integral, of dimension 5, Cohen-Macaulay and normal. Its singular locus is defined by $e = f = g = h = i = j = 0$.*

Proof. Let us consider $j' := j + ek$. A is isomorphic to $A' \otimes \Bbbk[k]$, where

$$A' := \Bbbk[e, f, g, h, i, j']/(fh - gi, ei - fj', eh - gj').$$

This ring is the algebra of functions on the variety of matrices

$$\begin{pmatrix} h & i & j' \\ g & f & e \end{pmatrix}$$

of rank at most 1, which is the cone of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$. This variety is well known to be integral, Cohen-Macaulay and normal, the vertex of the cone (defined by $e = f = g = h = i = j' = 0$) being its unique singularity (see *e.g.* [BV88, 2.8, 2.11]). The lemma follows. □

In particular, M is integral. It contains $S_{\{\alpha,\beta\}}|_{(B/B)\times(U^+B/B)}$ (the equations are satisfied on a dense open subset of $S_{\{\alpha,\beta\}}|_{(B/B)\times(U^+B/B)}$, hence everywhere), which has the same dimension. Hence the two varieties coincide.

We deduce that $S_{\{\alpha,\beta\}}$ is normal and Cohen-Macaulay. This finishes the proof of Proposition 3.3.1. □

3.4 End of the proof

We denote by $\Psi_{(s_\alpha,s_\beta,s_\alpha)} : \widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)} \to S_{\{\alpha,\beta\}}$ the morphism constructed above (it is the restriction of $p_{1,4}$), and similarly with α and β interchanged..

Proposition 3.4.1. *We have*

$$R(\Psi_{(s_\alpha,s_\beta,s_\alpha)})_*(\mathcal{O}_{\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}}) \cong \mathcal{O}_{S_{\{\alpha,\beta\}}},$$

and similarly with α and β interchanged.

Proof. First we prove that $R^i(\Psi_{(s_\alpha,s_\beta,s_\alpha)})_*(\mathcal{O}_{\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}}) = 0$ for $i \geq 1$. The argument for this is adapted from [BK04, 3.2.1]. Since the fibers of $\Psi_{(s_\alpha,s_\beta,s_\alpha)}$ are of dimension at most

1, by [Har77, III.11.2] we have $R^i(\Psi_{(s_\alpha,s_\beta,s_\alpha)})_* = 0$ for $i \geq 2$. Hence we only have to prove the equality $R^1(\Psi_{(s_\alpha,s_\beta,s_\alpha)})_*(\mathcal{O}_{\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}}) = 0$. The following diagram commutes:

$$\begin{tikzcd}
\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)} \arrow[r, hook, "i"] \arrow[d, "\Psi_{(s_\alpha,s_\beta,s_\alpha)}"'] & \mathfrak{g}^* \times \mathcal{Z}_{(s_\alpha,s_\beta,s_\alpha)} \arrow[d, "\mathrm{Id}\times\Phi_{(s_\alpha,s_\beta,s_\alpha)}"] \\
S_{\{\alpha,\beta\}} \arrow[r, hook, "j"] & \mathfrak{g}^* \times \mathcal{X}_{s_\alpha s_\beta s_\alpha}
\end{tikzcd}$$

where i and j are closed embeddings. Hence we only have to show the equality $R^1(\mathrm{Id} \times \Phi_{(s_\alpha,s_\beta,s_\alpha)})_*(i_*\mathcal{O}_{\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}}) = 0$. We have a surjection

$$\mathcal{O}_{\mathfrak{g}^* \times \mathcal{Z}_{(s_\alpha,s_\beta,s_\alpha)}} \twoheadrightarrow i_*\mathcal{O}_{\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}}.$$

As $R^2(\mathrm{Id} \times \Phi_{(s_\alpha,s_\beta,s_\alpha)})_* = 0$ (for the same reason as above), we obtain a surjection

$$R^1(\mathrm{Id} \times \Phi_{(s_\alpha,s_\beta,s_\alpha)})_*(\mathcal{O}_{\mathfrak{g}^* \times \mathcal{Z}_{(s_\alpha,s_\beta,s_\alpha)}}) \twoheadrightarrow R^1(\mathrm{Id} \times \Phi_{(s_\alpha,s_\beta,s_\alpha)})_*(i_*\mathcal{O}_{\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}}).$$

By the vanishing of higher direct images for Demazure resolutions (see *e.g.* [BK04, Theorem 3.3.4]), the object on the left hand side is zero. Hence $R^1(\mathrm{Id} \times \Phi_{(s_\alpha,s_\beta,s_\alpha)})_*(i_*\mathcal{O}_{\mathcal{Z}_{(s_\alpha,s_\beta,s_\alpha)}}) = 0$, as claimed.

Since $\Psi_{(s_\alpha,s_\beta,s_\alpha)}$ is proper and birational (because $\Phi_{(s_\alpha,s_\beta,s_\alpha)}$ is), and $S_{\{\alpha,\beta\}}$ is normal (by Proposition 3.3.1), one has $(\Psi_{(s_\alpha,s_\beta,s_\alpha)})_*(\mathcal{O}_{\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha)}}) \cong \mathcal{O}_{S_{\{\alpha,\beta\}}}$ by Zariski's Main Theorem. This proves the result. The assertion with α and β interchanged is obtained similarly. □

With this proposition the proof of the finite braid relation for the action on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$ (see Theorem 2.3.2) when α and β generate a root system of type $\mathbf{A}_2$ is complete.

4 Finite braid relations for type $\mathbf{B}_2$

Now we assume that α and β generate a root system of type $\mathbf{B}_2$. To fix notations, we assume that α is short and β is long. Then $\langle \alpha, \beta^\vee \rangle = -1$, $\langle \beta, \alpha^\vee \rangle = -2$. There exist structure constants $c, d \in \Bbbk^\times$ such that

$$\forall x, y \in \Bbbk, \quad (u_\alpha(x), u_\beta(y)) = u_{\alpha+\beta}(cxy)u_{2\alpha+\beta}(dx^2y)$$

(again, see [Spr98, 8.2.3]). Then, also,

$$\forall x, y \in \Bbbk, \quad (u_\beta(x), u_\alpha(y)) = u_{\alpha+\beta}(-cxy)u_{2\alpha+\beta}(-dxy^2).$$

Easy calculations yield the following formulae for the adjoint action of G on $\mathfrak{g}$:

$$\begin{array}{ll}
u_\alpha(x)\cdot e_\beta = e_\beta + cxe_{\alpha+\beta} + dx^2 e_{2\alpha+\beta}, & u_\alpha(x)\cdot h_\beta = h_\beta + xe_\alpha, \\
u_\alpha(x)\cdot e_{\alpha+\beta} = e_{\alpha+\beta} + 2\frac{d}{c}xe_{2\alpha+\beta}, & u_\beta(x)\cdot h_\alpha = h_\alpha + 2xe_\beta, \\
u_\beta(x)\cdot e_\alpha = e_\alpha - cxe_{\alpha+\beta}, & u_{\alpha+\beta}(x)\cdot h_\alpha = h_\alpha, \\
u_{\alpha+\beta}(x)\cdot e_\alpha = e_\alpha - 2\frac{d}{c}xe_{2\alpha+\beta}, & u_{\alpha+\beta}(x)\cdot h_\beta = h_\beta - xe_{\alpha+\beta}, \\
u_{\alpha+\beta}(x)\cdot e_{-\alpha} = e_{-\alpha} - \frac{2}{c}xe_\beta, & u_{2\alpha+\beta}(x)\cdot h_\alpha = h_\alpha - 2xe_{2\alpha+\beta}, \\
u_{\alpha+\beta}(x)\cdot e_{-\beta} = e_{-\beta} + \frac{1}{c}xe_\alpha - \frac{d}{c^2}x^2 e_{2\alpha+\beta}, & \\
u_{2\alpha+\beta}(x)\cdot e_{-\alpha} = e_{-\alpha} - \frac{2c}{d}xe_{\alpha+\beta}. &
\end{array}$$

We also have $h_{\alpha+\beta} = h_\alpha + 2h_\beta$, $h_{2\alpha+\beta} = h_\alpha + h_\beta$.

In this section we prove the finite braid relation for the simple roots α and β. The proof is very similar to the one in the previous section. We assume throughout the section that $\mathrm{char}(\Bbbk) \neq 2$.

4.1 Derived tensor product

Lemma 4.1.1. *There exist isomorphisms*

$$\mathcal{O}_{S_\alpha\times\widetilde{\mathfrak{g}}^3} \overset{L}{\otimes}_{\widetilde{\mathfrak{g}}^5} \mathcal{O}_{\widetilde{\mathfrak{g}}\times S_\beta\times\widetilde{\mathfrak{g}}^2} \overset{L}{\otimes}_{\widetilde{\mathfrak{g}}^5} \mathcal{O}_{\widetilde{\mathfrak{g}}^2\times S_\alpha\times\widetilde{\mathfrak{g}}} \overset{L}{\otimes}_{\widetilde{\mathfrak{g}}^5} \mathcal{O}_{\widetilde{\mathfrak{g}}^3\times S_\beta} \cong \mathcal{O}_{\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha,s_\beta)}};$$

$$\mathcal{O}_{S_\beta\times\widetilde{\mathfrak{g}}^3} \overset{L}{\otimes}_{\widetilde{\mathfrak{g}}^5} \mathcal{O}_{\widetilde{\mathfrak{g}}\times S_\alpha\times\widetilde{\mathfrak{g}}^2} \overset{L}{\otimes}_{\widetilde{\mathfrak{g}}^5} \mathcal{O}_{\widetilde{\mathfrak{g}}^2\times S_\beta\times\widetilde{\mathfrak{g}}} \overset{L}{\otimes}_{\widetilde{\mathfrak{g}}^5} \mathcal{O}_{\widetilde{\mathfrak{g}}^3\times S_\alpha} \cong \mathcal{O}_{\widetilde{\mathcal{Z}}_{(s_\beta,s_\alpha,s_\beta,s_\alpha)}}.$$

Moreover, the varieties $\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha,s_\beta)}$ and $\widetilde{\mathcal{Z}}_{(s_\beta,s_\alpha,s_\beta,s_\alpha)}$ are integral.

Proof. As for Lemma 3.1.1, we prove the result in the first case only, by computation of equations (the second case can be treated similarly). Let us choose an ordering of R^+ such that the last four roots are $2\alpha+\beta$, $\alpha+\beta$, β, α (in this order). Let $U^+_{(\alpha)}$, $U^+_{(\beta)}$, $U^+_{(\alpha,\beta)}$ be the product of the U_γ for $\gamma \in R^+ - \{\alpha\}$, $\gamma \in R^+ - \{\beta\}$, $\gamma \in R^+ - \{\alpha,\beta,\alpha+\beta,2\alpha+\beta\}$. Under the isomorphism $U^+ \cong \prod_{\gamma\in R^+} U_\gamma$, the restriction to U^+ of the projections $\pi_\alpha : \mathcal{B} \to \mathcal{P}_\alpha$, $\pi_\beta : \mathcal{B} \to \mathcal{P}_\beta$ become the morphisms $U^+_{(\alpha)} \times U_\alpha \to U^+_{(\alpha)}$ and

$$\left\{ \begin{array}{lcl} U^+_{(\alpha,\beta)} \times U_{2\alpha+\beta} \times U_{\alpha+\beta} \times U_\beta \times U_\alpha & \to & U^+_{(\alpha,\beta)} \times U_{2\alpha+\beta} \times U_{\alpha+\beta} \times U_\alpha \\ (u, u_{2\alpha+\beta}(x), u_{\alpha+\beta}(y), u_\beta(z), u_\alpha(t)) & \mapsto & (u, u_{2\alpha+\beta}(x-dt^2z), u_{\alpha+\beta}(y-czt), u_\alpha(t)) \end{array} \right. .$$

As coordinates we will use the $u^{(j)}$, $x^{(j)}$, $y^{(j)}$, $z^{(j)}$ and $t^{(j)}$ on the base ($j = 2,\dots,5$), and $e^{(j)}_\gamma$ ($\gamma \in R^+$), $h^{(j)}_\delta$ ($\delta \in \Phi$) in the fibers ($j = 1,\dots,5$).

In these coordinates, $(S_\alpha \times \widetilde{\mathfrak{g}}^3)|_{(B/B)\times(U^+B/B)^4} \subset (\widetilde{\mathfrak{g}}^5)|_{(B/B)\times(U^+B/B)^4}$ is defined by the equations

$$(*) \qquad u^{(2)} = 1,\ x^{(2)} = 0,\ y^{(2)} = 0,\ z^{(2)} = 0,\ e^{(1)}_\gamma = e^{(2)}_\gamma,\ h^{(1)}_\delta = h^{(2)}_\delta$$

for $\delta \in \Phi$, $\gamma \in R^+$, and

$$h^{(1)}_\alpha - t^{(2)} e^{(1)}_\alpha = 0. \tag{4.1.2}$$

Similarly, $(\widetilde{\mathfrak{g}} \times S_\beta \times \widetilde{\mathfrak{g}}^2)|_{(B/B)\times(U^+B/B)^4} \subset (\widetilde{\mathfrak{g}}^5)|_{(B/B)\times(U^+B/B)^4}$ is defined by the equations

$$(*') \qquad \left\{ \begin{array}{l} u^{(2)} = u^{(3)},\ x^{(2)} - d(t^{(2)})^2 z^{(2)} = x^{(3)} - d(t^{(3)})^2 z^{(3)},\ t^{(2)} = t^{(3)}, \\ y^{(2)} - cz^{(2)}t^{(2)} = y^{(3)} - cz^{(3)}t^{(3)},\ e_\gamma^{(2)} = e_\gamma^{(3)},\ h_\delta^{(2)} = h_\delta^{(3)}, \end{array} \right.$$

and $u^{(2)} \cdot u_{2\alpha+\beta}(x^{(2)} - d(t^{(2)})^2 z^{(2)}) \cdot u_{\alpha+\beta}(y^{(2)} - cz^{(2)}t^{(2)}) \cdot u_\alpha(t^{(2)}) \cdot (h_\beta^{(2)} - (z^{(2)} + z^{(3)}) e_\beta^{(2)}) = 0$, *i.e.*

$$\begin{aligned} u^{(2)} \cdot \big(h_\beta^{(2)} + t^{(2)} e_\alpha^{(2)} - (z^{(2)} + z^{(3)}) e_\beta^{(2)} + (-y^{(2)} - ct^{(2)} z^{(3)}) e_{\alpha+\beta}^{(2)} \\ + (-2\frac{d}{c} y^{(2)} t^{(2)} + d(t^{(2)})^2 (z^{(2)} - z^{(3)})) e_{2\alpha+\beta}^{(2)}\big) = 0. \end{aligned} \tag{4.1.3}$$

Next, $(\widetilde{\mathfrak{g}}^2 \times S_\alpha \times \widetilde{\mathfrak{g}})|_{(B/B)\times(U^+B/B)^4} \subset (\widetilde{\mathfrak{g}}^5)|_{(B/B)\times(U^+B/B)^4}$ is defined by the equations

$$(*'') \qquad \left\{ \begin{array}{l} u^{(3)} = u^{(4)},\ x^{(3)} = x^{(4)},\ y^{(3)} = y^{(4)}, \\ z^{(3)} = z^{(4)},\ e_\gamma^{(3)} = e_\gamma^{(4)},\ h_\delta^{(3)} = h_\delta^{(4)}, \end{array} \right.$$

and $u^{(3)} \cdot u_{2\alpha+\beta}(x^{(3)}) \cdot u_{\alpha+\beta}(y^{(3)}) \cdot u_\beta(z^{(3)}) \cdot (h_\alpha^{(3)} - (t^{(3)} + t^{(4)}) e_\alpha^{(3)}) = 0$, *i.e.*

$$\begin{aligned} u^{(3)} \cdot \big(h_\alpha^{(3)} - (t^{(3)} + t^{(4)}) e_\alpha^{(3)} + 2z^{(3)} e_\beta^{(3)} + cz^{(3)} (t^{(3)} + t^{(4)}) e_{\alpha+\beta}^{(3)} \\ + (-2x^{(3)} + 2\frac{d}{c} y^{(3)} (t^{(3)} + t^{(4)}) e_{2\alpha+\beta})\big) = 0. \end{aligned} \tag{4.1.4}$$

Finally, $(\widetilde{\mathfrak{g}}^3 \times S_\beta)|_{(B/B)\times(U^+B/B)^4} \subset (\widetilde{\mathfrak{g}}^5)|_{(B/B)\times(U^+B/B)^4}$ is defined by the equations

$$(*''') \qquad \left\{ \begin{array}{l} u^{(4)} = u^{(5)},\ x^{(4)} - d(t^{(4)})^2 z^{(4)} = x^{(5)} - d(t^{(5)})^2 z^{(5)},\ t^{(4)} = t^{(5)}, \\ y^{(4)} - cz^{(4)}t^{(4)} = y^{(5)} - cz^{(5)}t^{(5)},\ e_\gamma^{(4)} = e_\gamma^{(5)},\ h_\delta^{(4)} = h_\delta^{(5)}, \end{array} \right.$$

and $u^{(4)} \cdot u_{2\alpha+\beta}(x^{(4)} - d(t^{(4)})^2 z^{(4)}) \cdot u_{\alpha+\beta}(y^{(4)} - cz^{(4)}t^{(4)}) \cdot u_\alpha(t^{(4)}) \cdot (h_\beta^{(4)} - (z^{(4)} + z^{(5)}) e_\beta^{(4)}) = 0$, *i.e.*

$$\begin{aligned} u^{(4)} \cdot \big(h_\beta^{(4)} + t^{(4)} e_\alpha^{(4)} - (z^{(4)} + z^{(5)}) e_\beta^{(4)} + (-y^{(4)} - ct^{(4)} z^{(5)}) e_{\alpha+\beta}^{(4)} \\ + (-2\frac{d}{c} y^{(4)} t^{(4)} + d(t^{(4)})^2 (z^{(4)} - z^{(5)})) e_{2\alpha+\beta}^{(4)}\big) = 0. \end{aligned} \tag{4.1.5}$$

As in the proof of Lemma 3.1.1, we have to show that the union of these equations forms a regular sequence. The equations $(*)$ to $(*''')$ allow us to eliminate the coordinates $u^{(j)}$, $x^{(2)}$, $y^{(2)}$, $z^{(2)}$, $x^{(3)}$, $y^{(3)}$, $t^{(3)}$, $x^{(4)}$, $y^{(4)}$, $z^{(4)}$, $x^{(5)}$, $y^{(5)}$, $t^{(5)}$, and to identify the coordinates in the fibers, which we will denote by e_γ and h_δ. Then, equations (4.1.2) and (4.1.3) allow to eliminate h_α and h_β. With these simplifications, equations (4.1.4) and (4.1.5) become

$$-t^{(4)} e_\alpha + 2z^{(3)} e_\beta + cz^{(3)} (t^{(2)} + t^{(4)}) e_{\alpha+\beta} + 2dz^{(3)} t^{(2)} t^{(4)} e_{2\alpha+\beta} = 0, \tag{4.1.6}$$

$$(t^{(4)}-t^{(2)})e_\alpha - z^{(5)}e_\beta - ct^{(4)}z^{(5)}e_{\alpha+\beta} + d(z^{(3)}(t^{(2)}-t^{(4)})^2 - (t^{(4)})^2 z^{(5)})e_{2\alpha+\beta} = 0. \quad (4.1.7)$$

Let us denote by P the polynomial of (4.1.6), and by Q the polynomial of (4.1.7). Then P and Q are irreducible and distinct. Hence they form a regular sequence in $\Bbbk[z^{(3)}, z^{(5)}, t^{(2)}, t^{(4)}, e_\gamma, \gamma \in R^+, h_\delta, \delta \in \Phi - \{\alpha,\beta\}]$. This proves that the tensor product we are considering is indeed concentrated in degree 0, and that the quotient ring $\Bbbk[z^{(3)}, z^{(5)}, t^{(2)}, t^{(4)}, e_\gamma, h_\delta]/(P,Q)$ is Cohen-Macaulay (see again [BH93, 2.1.3]). We prove in the next lemma that this ring is an integral domain. We deduce, as in the case of $\mathbf{A}_2$, that $\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha,s_\beta)}$ is an integral scheme. □

Lemma 4.1.8. *$\Bbbk[z^{(3)}, z^{(5)}, t^{(2)}, t^{(4)}, e_\gamma, h_\delta]/(P,Q)$ is an integral domain.*

Proof. First, let us prove that the closed subvariety N of $\Bbbk^{\dim(\mathfrak{g}/\mathfrak{n})+2}$ defined by P and Q is irreducible. The restriction of this subvariety to the open set defined by $t^{(4)} \neq 0$ is irreducible (indeed, on this open set P gives e_α as a polynomial in the other coordinates and $(t^{(4)})^{-1}$, and replacing in Q we still obtain an irreducible polynomial). Similarly for the intersections with the open set defined by $z^{(3)} \neq 0$, and with the open set defined by $z^{(5)} \neq 0$. Now N is isomorphic to the closure of its intersection with the open set $\{t^{(4)} \neq 0\} \cup \{z^{(3)} \neq 0\} \cup \{z^{(5)} \neq 0\}$ (indeed, if $t^{(4)} = z^{(3)} = 0$, P is zero, and $Q = -t^{(2)}e_\alpha - z^{(5)}e_\beta$ is an irreducible polynomial, whose variety of zeros intersect the open set $\{z^{(5)} \neq 0\}$). This intersection is irreducible (it is the union of three irreducible open sets, each one intersecting each other one). Hence N is irreducible.

Now we have to show that the ring $\Bbbk[z^{(3)}, z^{(5)}, t^{(2)}, t^{(4)}, e_\gamma, h_\delta]/(P,Q)$ is reduced, *i.e.* that it satisfies properties (R_0) and (S_1) (see [Mat80, p. 125]). As we have seen that it is Cohen-Macaulay, and that the corresponding scheme is irreducible, we only have to prove that it is regular at some point. But it is clearly regular at the point defined by $t^{(2)} = t^{(4)} = 1$, $z^{(3)} = 0$, $z^{(5)} = 1$, $e_\alpha = e_\beta = e_{\alpha+\beta} = e_{2\alpha+\beta} = 0$ (consider the partial differentials of P and Q with respect to e_α and e_β). □

4.2 Determination of the image

As in 3.2, we have to identify the images of $\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha,s_\beta)}$ and $\widetilde{\mathcal{Z}}_{(s_\beta,s_\alpha,s_\beta,s_\alpha)}$ under $p_{1,5} : \widetilde{\mathfrak{g}}^5 \to \widetilde{\mathfrak{g}}^2$ (these are closed subvarieties of $\widetilde{\mathfrak{g}}^2$), and observe that they coincide. We only indicate the computations for the first case, the second one being similar. By G-equivariance we only have to compute the fibers of this image over the points $(B/B, wB/B)$ for w in the subgroup of W generated by s_α and s_β. In this case the Demazure resolution $\Phi_{(s_\alpha,s_\beta,s_\alpha,s_\beta)}$ is an isomorphism over the complement of $\mathcal{X}_{s_\alpha s_\beta}$. This gives the condition on $X \in \mathfrak{g}^*$ for the point $(X, B/B, wB/B)$ to be in $p_{1,5}(\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha,s_\beta)})$ in cases (i) to (iv).

(i) Fiber over $(B/B, s_\alpha s_\beta s_\alpha s_\beta B/B)$: $X_{|\mathfrak{n} \oplus \Bbbk e_\alpha \oplus \Bbbk e_\beta \oplus \Bbbk e_{\alpha+\beta} \oplus \Bbbk e_{2\alpha+\beta}} = 0$.

(ii) Fiber over $(B/B, s_\alpha s_\beta s_\alpha B/B)$: $X_{|\mathfrak{n} \oplus \Bbbk e_\alpha \oplus \Bbbk e_{\alpha+\beta} \oplus \Bbbk e_{2\alpha+\beta} \oplus \Bbbk h_\beta} = 0$.

(iii) Fiber over $(B/B, s_\beta s_\alpha s_\beta B/B)$: $X_{|\mathfrak{n} \oplus \Bbbk e_\beta \oplus \Bbbk e_{\alpha+\beta} \oplus \Bbbk e_{2\alpha+\beta} \oplus \Bbbk h_\alpha} = 0$.

(iv) Fiber over $(B/B, s_\beta s_\alpha B/B)$: $X_{|\mathfrak{n} \oplus \Bbbk e_\beta \oplus \Bbbk e_{\alpha+\beta} \oplus \Bbbk h_\alpha \oplus \Bbbk h_\beta} = 0$.

(v) Fiber over $(B/B, s_\alpha s_\beta B/B)$: the fiber of $\Phi_{(s_\alpha,s_\beta,s_\alpha,s_\beta)}$ is isomorphic to two copies of $\mathbb{P}^1_{\Bbbk}$ with one common point. It contains, on the one hand, the points of the form $(B/B, s_\alpha B/B, s_\alpha gB/B, s_\alpha gB/B, s_\alpha s_\beta B/B)$ for $g \in P_\beta$ and, on the other hand, the points $(B/B, gB/B, gB/B, s_\alpha B/B, s_\alpha s_\beta B/B)$ for $g \in P_\alpha$. One verifies that the conditions on X corresponding to each of these points are the same, namely

$$X_{|\mathfrak{n}\oplus\Bbbk e_\alpha\oplus\Bbbk e_{2\alpha+\beta}\oplus\Bbbk h_\alpha\oplus\Bbbk h_\beta} = 0.$$

(vi) Fiber over $(B/B, s_\alpha B/B)$: the fiber of the Demazure resolution is formed by the points $(B/B, gB/B, gB/B, s_\alpha B/B, s_\alpha B/B)$ for $g \in P_\alpha$ and the points of the form $(B/B, s_\alpha B/B, s_\alpha gB/B, s_\alpha gB/B, s_\alpha B/B)$ for $g \in P_\beta$. Let us compute the conditions on X corresponding to the each of these points. First we consider a point of the form $(B/B, gB/B, gB/B, s_\alpha B/B, s_\alpha B/B)$ for some $g \in P_\alpha$. If $g \in s_\alpha B$, the condition is to vanish on $\mathfrak{n}$, e_α, $s_\alpha \cdot h_\beta = h_\alpha + h_\beta$ and $s_\alpha \cdot h_\alpha = -h_\alpha$, *i.e.* $X_{|\mathfrak{n}\oplus\Bbbk e_\alpha\oplus\Bbbk h_\alpha\oplus\Bbbk h_\beta} = 0$. If $g = u_\alpha(\epsilon)$ for some $\epsilon \in \Bbbk$, then the condition is to vanish on $h_\alpha - \epsilon e_\alpha$, $u_\alpha(\epsilon)\cdot h_\beta = h_\beta + \epsilon e_\alpha$, e_α and $s_\alpha \cdot h_\beta = h_\alpha + h_\beta$, *i.e.* the same condition. Now, let us consider the points $(B/B, s_\alpha B/B, s_\alpha gB/B, s_\alpha gB/B, s_\alpha B/B)$ for $g \in P_\beta$. If $g \in s_\beta B$, then the condition is to vanish on e_α, $e_{2\alpha+\beta}$ and $s_\alpha s_\beta \cdot h_\alpha = h_{\alpha+\beta}$. If $g = u_\beta(\epsilon)$, the condition is to vanish on e_α, $s_\alpha \cdot (h_\beta - \epsilon e_\beta)$ and $s_\alpha u_\beta(\epsilon)\cdot h_\alpha = s_\alpha \cdot (h_\alpha + 2\epsilon e_\beta)$, *i.e.* on e_α, $s_\alpha \cdot (h_\beta - \epsilon e_\beta)$ and $s_\alpha \cdot (h_\alpha + 2h_\beta) = s_\alpha \cdot h_{\alpha+\beta} = h_{\alpha+\beta}$. As in 3.2 (vi), the condition on X for the point $(X, B/B, s_\alpha B/B)$ to be in the image of $p_{1,5}$ is finally

$$X_{|\mathfrak{n}\oplus\Bbbk e_\alpha\oplus\Bbbk h_{\alpha+\beta}} = 0.$$

(vii) Fiber over $(B/B, s_\beta B/B)$: Similarly, the condition is

$$X_{|\mathfrak{n}\oplus\Bbbk e_\beta\oplus\Bbbk h_{2\alpha+\beta}} = 0.$$

(viii) Fiber over $(B/B, B/B)$: the fiber of the Demazure resolution is given on the one hand by the $(B/B, gB/B, gB/B, B/B, B/B)$ for $g \in P_\alpha$ and on the other hand by the $(B/B, B/B, gB/B, gB/B, B/B)$ for $g \in P_\beta$. In the first case, if $g \in B$ then the corresponding condition of X is to vanish on $\mathfrak{n}$, h_α and h_β. If $g \notin B$, then the condition is to vanish on $\mathfrak{n}$, e_α, h_α and h_β. The situation is similar in the second case. Hence the condition on X for $(X, B/B, B/B)$ to be in the image is

$$X_{|\mathfrak{n}\oplus\Bbbk h_\alpha\oplus\Bbbk h_\beta} = 0.$$

It follows from these computations and the similar ones with α and β interchanged (computing $p_{1,5}(\widetilde{\mathcal{Z}}_{(s_\beta,s_\alpha,s_\beta,s_\alpha)})$ instead of $p_{1,5}(\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha,s_\beta)})$ amounts to replacing α by β, β by α, $\alpha+\beta$ by $\beta+2\alpha$, and $\beta+2\alpha$ by $\alpha+\beta$) that the images under $p_{1,5}$ of $\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha,s_\beta)}$ and $\widetilde{\mathcal{Z}}_{(s_\beta,s_\alpha,s_\beta,s_\alpha)}$ coincide. We let $S_{\{\alpha,\beta\}}$ be this image.

4.3 Normality of $S_{\{\alpha,\beta\}}$

Proposition 4.3.1. *The variety $S_{\{\alpha,\beta\}}$ is integral and normal.*

Proof.[7] Let us define $\gamma := \alpha+\beta$, $\delta := 2\alpha+\beta$. As for type $\mathbf{A}_2$, we already know that $S_{\{\alpha,\beta\}}$

[7] This proof is a simplification of an earlier one due to Patrick Polo.

is integral, and we only have to consider the situation over $(B/B) \times (U_\delta U_\gamma U_\beta U_\alpha B/B)$. In this proof we consider $S_{\{\alpha,\beta\}}$ as the image of $\widetilde{\mathcal{Z}}_{(s_\beta,s_\alpha,s_\beta,s_\alpha)}$.

Let

$$u = u_\delta(x_\delta)u_\gamma(x_\gamma)u_\beta(x_\beta)u_\alpha(x_\alpha) \in U_\delta U_\gamma U_\beta U_\alpha,$$

with $x_\alpha x_\beta x_\gamma x_\delta \neq 0$, $x_\gamma x_\beta - \frac{d}{c^2}x_\gamma^2 \neq 0$ and $x_\alpha x_\gamma - \frac{c}{d}x_\delta \neq 0$. We have

$$u = u_\beta(t)u_\alpha(z)u_\beta(y)u_\alpha(x)$$

with $x_\alpha = x+z$, $x_\beta = y+t$, $x_\gamma = cyz$, $x_\delta = dyz^2$ (here $xyzt \neq 0$). Then if $(X, B/B, uB/B)$ is in $S_{\{\alpha,\beta\}}$, X must vanish on

$$h_\beta - te_\beta. \tag{4.3.2}$$

It also vanishes on $u_\beta(t)u_\alpha(z) \cdot e_{-\alpha}$, hence on $h_\alpha + 2te_\beta - ze_\alpha + czte_\gamma$. Adding two times (4.3.2), one obtains

$$h_\gamma - ze_\alpha + czte_\gamma. \tag{4.3.3}$$

Further, X must vanish on $u_\beta(t)u_\alpha(z)u_\beta(y) \cdot e_{-\beta}$, hence on $u_\beta(t)u_\alpha(z) \cdot (h_\beta - ye_\beta)$. Substracting (4.3.2), one obtains

$$ze_\alpha - (y+t)e_\beta - cz(y+t)e_\gamma - dyz^2e_\delta. \tag{4.3.4}$$

Finally, X vanishes on $u_\beta(t)u_\alpha(z)u_\beta(y)u_\alpha(x) \cdot e_{-\alpha}$, hence on $u_\beta(t)u_\alpha(z)u_\beta(y) \cdot (h_\alpha - xe_\alpha)$. Substracting $u_\beta(t) \cdot (h_\alpha - ze_\alpha)$, one obtains

$$-(x+z)e_\alpha + 2ye_\beta + c((y+t)(x+z) + yz)e_\gamma + 2dyz(x+z)e_\delta. \tag{4.3.5}$$

Let us transform our equations (4.3.2) to 4.3.5 to obtain equations in x_α, x_β, x_γ, x_δ. Substracting (4.3.5) from two times 4.3.2, one obtains

$$2h_\beta + x_\alpha e_\alpha - 2x_\beta e_\beta - (cx_\alpha x_\beta + x_\gamma)e_\gamma - \frac{2d}{c}x_\alpha x_\gamma e_\delta. \tag{4.3.6}$$

Similarly, adding (4.3.3) and (4.3.4), one obtains

$$h_\gamma - x_\beta e_\beta - x_\gamma e_\gamma - x_\delta e_\delta. \tag{4.3.7}$$

Then, one verifies that $(x+z)$ times (4.3.4) plus z times (4.3.5), and $2y$ times (4.3.4) plus v times (4.3.5) give respectively

$$(\frac{2}{c}x_\gamma - x_\alpha x_\beta)e_\beta + \frac{c}{d}x_\delta e_\gamma + x_\alpha x_\delta e_\delta, \tag{4.3.8}$$

$$(\frac{2}{c}x_\gamma - x_\alpha x_\beta)e_\alpha + x_\beta(cx_\alpha x_\beta - x_\gamma)e_\gamma + \frac{2d}{c}x_\gamma(x_\alpha x_\beta - \frac{1}{c}x_\gamma)e_\delta. \tag{4.3.9}$$

Finally, x_γ times (4.3.4) gives

$$\frac{c}{d}x_\delta e_\alpha - x_\beta x_\gamma e_\beta - \frac{c^2}{d}x_\beta x_\delta e_\gamma - x_\gamma x_\delta e_\delta. \tag{4.3.10}$$

Equations (4.3.6) and 4.3.7 express h_β, h_γ in terms of the other variables. We denote by E, F and G the polynomials of (4.3.8), (4.3.9) and (4.3.10).

Now we can finish the proof exactly as in the case of $\mathbf{A}_2$. In the next lemma we show that the scheme defined by E, F and G is normal and integral. Moreover it contains $S_{\{\alpha,\beta\}}|_{(B/B)\times(U^+B/B)}$ as a closed subvariety, and has the same dimension. Hence the two varieties coincide. □

Lemma 4.3.11. *The ring*

$$A := \Bbbk[x_\alpha, x_\beta, x_\gamma, x_\delta, e_\alpha, e_\beta, e_\gamma, e_\delta]/(E, F, G)$$

is a normal domain.

Proof. Let us forget about the previous notations x, y, z and t. Now we define $x = x_\alpha$, $y = -x_\beta$, $z = \frac{2}{c}x_\gamma - x_\alpha x_\beta$, $t = -x_\delta$, $f = \frac{2}{c}(\frac{c}{d}e_\alpha - \frac{c^2}{d}x_\beta e_\gamma - x_\gamma e_\delta)$, $g = e_\beta$, $h = \frac{c}{d}e_\gamma + x_\alpha e_\delta$, $i = e_\delta$. Then we have $A \cong A'[i]$, where

$$A' := \Bbbk[x, y, z, t, f, g, h]/(zg - th, zf - (z - xy)h, y(z - xy)g - tf).$$

Let us first show that the closed subvariety of $\Bbbk^7$ corresponding to A', denoted by M, is irreducible. The restriction of M to the open set $\{t \neq 0\}$ is defined by the equations $h = zg/t$ and $f = y(z-xy)g/t$. Hence it is irreducible. Similarly for the open sets $\{z \neq 0\}$ and $\{f \neq 0\}$. These open sets intersect each other in M. Hence the restriction of M to $\{t \neq 0\} \cup \{z \neq 0\} \cup \{f \neq 0\}$ is also irreducible. As M is the closure of this restriction (indeed, if $z = t = 0$, the condition $(x, y, z, t, f, g, h) \in V$ does not depend on f), it is irreducible.

Now we show that A is normal (hence also reduced). We will use the following lemma (see [BV88, 16.24]):

Lemma 4.3.12. *Let S be a noetherian ring, and $y \in S$ which is not a zero divisor. Assume that $S/(y)$ is reduced and $S[y^{-1}]$ is normal. Then S is normal.*

Let us apply the lemma to $S = A'$ and our element y. It is clear that y is not nilpotent (it is not zero on M). Since M is irreducible, y is not a zero-divisor. Now $A'/(y)$ is isomorphic to

$$\Bbbk[x, z, t, f, g, h]/I$$

where $I = (zg - th, zf, ft)$. This ideal is the intersection of the prime ideals (z, t) and $(f, zg - th)$ of $\Bbbk[x, z, t, f, g, h]$, hence it is reduced.

Consider the ring $A'[y^{-1}]$. Using the change of coordinates $f' = f/(y^2)$ and $x' = -x + (z/y)$, it is isomorphic to

$$\big(\Bbbk[x', z, t, f', g, h]/(zg - th, x'g - f't, zf' - hx')\big)[y, y^{-1}].$$

As in the proof of Lemma 3.3.9, this ring is normal. This concludes the proof of Lemma 4.3.11. □

Remark 4.3.13. As in type $\mathbf{A}_2$, one can show that $S_{\{\alpha,\beta\}}$ is Cohen-Macaulay. As our proof is long and not needed here, we omit it.

4.4 End of the proof

Now, exactly as in Proposition 3.4.1, one proves that

$$R(\Psi_{(s_\alpha,s_\beta,s_\alpha,s_\beta)})_*(\mathcal{O}_{\widetilde{\mathcal{Z}}_{(s_\alpha,s_\beta,s_\alpha,s_\beta)}}) = \mathcal{O}_{S_{\{\alpha,\beta\}}},$$

and similarly with α and β interchanged. This finishes the proof of the finite braid relations in type $\mathbf{B}_2$, hence also of the assertions of Theorem 2.3.2 concerning the action of B'_{aff} on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$, under the first assumptions.

In sections 5 to 7 we admit the theorem under the second assumptions (it will be proved in section 8).

5 Restriction to $\widetilde{\mathcal{N}}$

Now we will derive the assertions of Theorem 2.3.2 concerning the action of B'_{aff} on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$. We keep the notation and assumptions as before.

Let $i : \widetilde{\mathcal{N}} \hookrightarrow \widetilde{\mathfrak{g}}$ denote the closed embedding. For $\alpha \in \Phi$, we recall that $S'_\alpha := S_\alpha \cap (\widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}})$, and that Γ_i denotes the graph of i, a closed subvariety of $\widetilde{\mathcal{N}} \times \widetilde{\mathfrak{g}}$. First, relations (2), to (4) of Theorem 1.1.3 for the action on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$ can be proved exactly as for the action on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$ (see 2.5). Now we prove relations (1).

Lemma 5.1. *The tensor product $\mathcal{O}_{\widetilde{\mathcal{N}}\times\widetilde{\mathfrak{g}}} \overset{L}{\otimes}_{\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}}} \mathcal{O}_{S_\alpha}$ is concentrated in degree 0, and is isomorphic to $(i\times i)_*\mathcal{O}_{S'_\alpha}$.*

Proof. As in the proof of Proposition 2.4.2, we only have to consider the situation over $(B/B)\times(U^+B/B) \cong U^+$. We use the isomorphism $U^+ \cong U^+_{(\alpha)} \times U_\alpha$, and choose coordinates u on $U^+_{(\alpha)}$, t on U_α. On the fiber we use coordinates $e^{(j)}_\gamma$, $h^{(j)}_\delta$ $(j=1,2)$.

Then $(\widetilde{\mathcal{N}}\times\widetilde{\mathfrak{g}})|_{(B/B)\times(U^+B/B)}$ is defined by the equations $h^{(1)}_\delta = 0$ $(\delta\in\Phi)$, and S_α by $e^{(1)}_\gamma = e^{(2)}_\gamma$, $h^{(1)}_\delta = h^{(2)}_\delta$, $u=1$ and $h^{(1)}_\alpha - te^{(1)}_\alpha$. The union of these equations forms a regular sequence, which proves the result. □

Remark 5.2. These computations show that S'_α is reduced. It is not irreducible (see 7.1 for details).

Corollary 5.3. *There exist isomorphisms in $\mathcal{D}^b_{\mathrm{prop}}\mathrm{Coh}(\widetilde{\mathcal{N}}\times\widetilde{\mathfrak{g}})$:*

$$\begin{aligned} \mathcal{O}_{\Gamma_i} * \mathcal{O}_{S'_\alpha} &\cong \mathcal{O}_{S_\alpha} * \mathcal{O}_{\Gamma_i}; \\ \mathcal{O}_{\Gamma_i} * \mathcal{O}_{S'_\alpha}(\rho-\alpha,-\rho) &\cong \mathcal{O}_{S_\alpha}(\rho-\alpha,-\rho) * \mathcal{O}_{\Gamma_i}. \end{aligned}$$

Proof. We only prove the first isomorphism; the second one can be obtained similarly. It follows from Lemma 2.1.3 that $\mathcal{O}_{\Gamma_i} * \mathcal{O}_{S'_\alpha} \cong (\mathrm{Id}_{\widetilde{\mathcal{N}}} \times i)_*\mathcal{O}_{S'_\alpha}$. Hence we only have to prove that $\mathcal{O}_{S_\alpha} * \mathcal{O}_{\Gamma_i} \cong (\mathrm{Id}_{\widetilde{\mathcal{N}}} \times i)_*\mathcal{O}_{S'_\alpha}$.

Let $p_{a,b}$ denote the projections from $\widetilde{\mathcal{N}}\times\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}}$ to $\widetilde{\mathcal{N}}\times\widetilde{\mathfrak{g}}$ or $\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}}$, and $\Delta : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}}$ denote the diagonal embedding. Then by definition $\mathcal{O}_{S_\alpha} * \mathcal{O}_{\Gamma_i} = R(p_{1,3})_*(p^*_{1,2}\mathcal{O}_{\Gamma_i} \overset{L}{\otimes} p^*_{2,3}\mathcal{O}_{S_\alpha})$.

But $p_{1,2}^*\mathcal{O}_{\Gamma_i} \cong (\mathrm{Id}_{\widetilde{\mathcal{N}}} \times i \times \mathrm{Id}_{\widetilde{\mathfrak{g}}})_*(\Delta \times \mathrm{Id}_{\widetilde{\mathfrak{g}}})_*\mathcal{O}_{\widetilde{\mathcal{N}}\times\widetilde{\mathfrak{g}}}$. The result follows, using the projection formula and the preceding lemma, which implies that $L(i\times\mathrm{Id}_{\widetilde{\mathfrak{g}}})^*\mathcal{O}_{S_\alpha} \cong (\mathrm{Id}_{\widetilde{\mathcal{N}}}\times i)_*\mathcal{O}_{S'_\alpha}$. □

Corollary 5.4. *The finite braid relations* (i.e. *relations* 1 *of Theorem* 1.1.3) *are satisfied by the kernels* $\mathcal{O}_{S'_\alpha}$ $(\alpha \in \Phi)$.

Proof. First, let us prove an analogue of Proposition 2.4.2 for the kernels $\mathcal{O}_{S'_\alpha}$, *i.e.* that we have

$$(\dagger)\qquad \mathcal{O}_{S'_\alpha} * \big(\mathcal{O}_{S'_\alpha}(\rho-\alpha,-\rho)\big) \cong \Delta_*\mathcal{O}_{\widetilde{\mathcal{N}}} \cong \big(\mathcal{O}_{S'_\alpha}(\rho-\alpha,-\rho)\big) * \mathcal{O}_{S'_\alpha}.$$

Multiplying the equality $\mathcal{O}_{S_\alpha} * (\mathcal{O}_{S_\alpha}(\rho-\alpha,-\rho)) = \Delta_*\mathcal{O}_{\widetilde{\mathfrak{g}}}$ with $\mathcal{O}_{\Gamma_i}$ on the right, and using Lemma 2.1.3 and Corollary 5.3, one obtains

$$(\mathrm{Id}_{\widetilde{\mathcal{N}}}\times i)_*\big(\mathcal{O}_{S'_\alpha} * \big(\mathcal{O}_{S'_\alpha}(\rho-\alpha,-\rho)\big)\big) \cong (\mathrm{Id}_{\widetilde{\mathcal{N}}}\times i)_*(\Delta_*\mathcal{O}_{\widetilde{\mathcal{N}}}).$$

It follows that the complex of sheaves $\mathcal{O}_{S'_\alpha} * (\mathcal{O}_{S'_\alpha}(\rho-\alpha,-\rho))$ has its cohomology concentrated in degree 0, *i.e.* is isomorphic to a coherent sheaf on $\widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}}$. Then, as $(\mathrm{Id}_{\widetilde{\mathcal{N}}}\times i)_* : \mathrm{Coh}(\widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}}) \to \mathrm{Coh}(\widetilde{\mathcal{N}}\times\widetilde{\mathfrak{g}})$ has a left inverse $(\mathrm{Id}_{\widetilde{\mathcal{N}}}\times i)^*$, we deduce the first isomorphism in (†). The second one can be proved similarly.

Now, let us prove that the braid relations are satisfied. To fix notations, assume that α and β are simple roots generating a root system of type $\mathbf{A}_2$ (the other cases can be treated similarly). We have to prove that $\mathcal{O}_{S'_\alpha} * \mathcal{O}_{S'_\beta} * \mathcal{O}_{S'_\alpha} \cong \mathcal{O}_{S'_\beta} * \mathcal{O}_{S'_\alpha} * \mathcal{O}_{S'_\beta}$. By (†), this is equivalent to

$$\mathcal{O}_{S'_\beta}(\rho-\beta,-\rho) * \mathcal{O}_{S'_\alpha}(\rho-\alpha,-\rho) * \mathcal{O}_{S'_\beta}(\rho-\beta,-\rho) * \mathcal{O}_{S'_\alpha} * \mathcal{O}_{S'_\beta} * \mathcal{O}_{S'_\alpha} \cong \Delta_*\mathcal{O}_{\widetilde{\mathcal{N}}}.$$

But we know (see section 3) that

$$\mathcal{O}_{S_\beta}(\rho-\beta,-\rho) * \mathcal{O}_{S_\alpha}(\rho-\alpha,-\rho) * \mathcal{O}_{S_\beta}(\rho-\beta,-\rho) * \mathcal{O}_{S_\alpha} * \mathcal{O}_{S_\beta} * \mathcal{O}_{S_\alpha} \cong \Delta_*\mathcal{O}_{\widetilde{\mathfrak{g}}}.$$

Hence we can use the same argument as in the first part of this proof. □

Remark 5.5. The restriction of this action to B_{aff}, for R of type $\mathbf{A}$, was also considered in [KT07]. There, it was proved to have some nice properties.

6 Relation to localization in positive characteristic

In this section we show that the action of B'_{aff} on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$ we have constructed above, or rather the similar action on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})$ (for $\widetilde{\mathfrak{g}}^{(1)}$ the Frobenius twist of $\widetilde{\mathfrak{g}}$, see [BMR08, 1.1.1]), extends the action on $\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}_\chi}(\widetilde{\mathfrak{g}}^{(1)})$ constructed in [BMR06] using representation theory of Lie algebras and $\mathcal{D}$-modules in positive characteristic.

In 6.1 and 6.2, $\Bbbk$ is an arbitrary algebraically closed field. In 6.3 we assume $\mathrm{char}(\Bbbk) > h$ for h the Coxeter number of G. We use the same notation as above.

6.1 The reflection functors

Let us fix a simple root $\alpha \in \Phi$. In this subsection we study the functor $L(\widetilde{\pi}_\alpha^{(1)})^* \circ R(\widetilde{\pi}_\alpha^{(1)})_*$. To simplify notations, we forget about the Frobenius twists; the "twisted versions" of our results can be proved similarly. In this subsection and the next one, $\mathrm{char}(\Bbbk)$ is arbitrary.

We are in the situation of Lemma 2.1.2, with f being the morphism $\widetilde{\pi}_\alpha$. So $L(\widetilde{\pi}_\alpha)^* \circ R(\widetilde{\pi}_\alpha)_*$ is the convolution functor with kernel

$$R(p_{13})_*(\mathcal{O}_{\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}_\alpha \times \widetilde{\mathfrak{g}}} \overset{L}{\otimes}_{\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}_\alpha \times \widetilde{\mathfrak{g}}} \mathcal{O}_{\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}_\alpha \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}}).$$

The situation is particularly simple here, due to the following result:

Lemma 6.1.1. *The derived tensor product $\mathcal{O}_{\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}_\alpha \times \widetilde{\mathfrak{g}}} \overset{L}{\otimes}_{\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}_\alpha \times \widetilde{\mathfrak{g}}} \mathcal{O}_{\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}_\alpha \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}}$ is concentrated in degree 0[8]. It equals the sheaf of functions on the intersection $(\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}_\alpha \times \widetilde{\mathfrak{g}}) \cap (\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}_\alpha \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}})$. Moreover, this intersection is reduced.*

Proof. This proof is again similar to the proof of Proposition 2.4.2. For simplicity, in this proof we write P for P_α. We can restrict to the situation over $(B/B) \times (U^+P/P) \times (U^+B/B) \cong U^+_{(\alpha)} \times U^+$. We use the isomorphisms $\widetilde{\mathfrak{g}}|_{U^+B/B} \cong (\mathfrak{b}^+)^* \times U^+$ and $\widetilde{\mathfrak{g}}_\alpha|_{U^+P/P} \cong (\mathfrak{b}^+ \oplus \Bbbk e_{-\alpha})^* \times U^+_{(\alpha)}$ induced by restriction, and choose as usual coordinates $e_\gamma^{(i)}$, $h_\delta^{(i)}$ ($\gamma \in R^+$, $\delta \in \Phi$, $i \in \{1,2,3\}$) and $e_{-\alpha}^{(2)}$ in the fibers, $u^{(2)}$ and $u^{(3)}$ on $U^+_{(\alpha)}$, and t on U_α.

The equations of the first subvariety are $e_\gamma^{(1)} = e_\gamma^{(2)}$, $h_\delta^{(1)} = h_\delta^{(2)}$, $e_{-\alpha}^{(2)} = 0$ and $u^{(2)} = 1$. And the equations of the second variety are $e_\gamma^{(2)} = e_\gamma^{(3)}$, $h_\delta^{(2)} = h_\delta^{(3)}$, $u^{(2)} = u^{(3)}$ and $u^{(2)} \cdot u_\alpha(t) \cdot e_{-\alpha}^{(2)} = 0$, *i.e.* $u^{(2)} \cdot (e_{-\alpha}^{(2)} + th_\alpha^{(2)} - t^2 e_\alpha^{(2)}) = 0$.

It is clear that these equations form a regular sequence, and define a reduced scheme. This proves the lemma. □

The morphism $p_{1,3}$ restricts to an isomorphism from the intersection $(\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}_\alpha \times \widetilde{\mathfrak{g}}) \cap (\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}_\alpha \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}})$ to $\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}$. Hence we obtain, using Lemma 2.1.2:

Proposition 6.1.2. *There exists an isomorphism of functors*

$$L(\widetilde{\pi}_\alpha)^* \circ R(\widetilde{\pi}_\alpha)_* \cong F^{\mathcal{O}_{\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}}}_{\widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}}$$

for the closed subvariety $\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}} \subset \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$.

Moreover, under this isomorphism, the adjunction morphism $L(\widetilde{\pi}_\alpha)^* \circ R(\widetilde{\pi}_\alpha)_* \to \mathrm{Id}$ is induced by the restriction map $\mathcal{O}_{\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}} \to \Delta_* \mathcal{O}_{\widetilde{\mathfrak{g}}}$.

6.2 Intertwining functors

We have seen in 2.3 that $\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}} = G \times^B \mathscr{R}_\alpha$, and that the B-variety $\mathscr{R}_\alpha$ has two irreducible components, $\mathscr{D}_\alpha$ and $\mathscr{S}_\alpha$.

[8] As noticed by Michel Brion, this property is a general fact for morphisms between smooth varieties.

Lemma 6.2.1. *There exist exact sequences of B-equivariant quasi-coherent sheaves on $\mathfrak{g}^* \times P_\alpha/B$, where the surjections are restriction maps:*

$$\mathcal{O}_{\mathscr{D}_\alpha} \hookrightarrow \mathcal{O}_{\mathscr{R}_\alpha} \twoheadrightarrow \mathcal{O}_{\mathscr{S}_\alpha};$$
$$\mathcal{O}_{\mathscr{S}_\alpha}(-\rho) \otimes_{\Bbbk} \Bbbk_B(\rho-\alpha) \hookrightarrow \mathcal{O}_{\mathscr{R}_\alpha} \twoheadrightarrow \mathcal{O}_{\mathscr{D}_\alpha}.$$

Proof. : We use the same notation as in 2.3. In particular, recall the equations of $\mathscr{R}_\alpha$, $\mathscr{S}_\alpha$, $\mathscr{D}_\alpha$. On $U_\alpha B/B$, we have an exact sequence

$$\Bbbk[h_\delta, e_\gamma, t]/(t) \hookrightarrow \Bbbk[h_\delta, e_\gamma, t]/(t(h_\alpha - te_\alpha)) \twoheadrightarrow \Bbbk[h_\delta, e_\gamma, t]/(h_\alpha - te_\alpha)$$

where the first map is multiplication by $(h_\alpha - te_\alpha)$. Under the change of coordinates on $(U_\alpha B/B) \cap (n_\alpha U_\alpha B/B)$ (given by $t \mapsto -\frac{1}{t}$), $h_\alpha - te_\alpha$ is sent to $h_\alpha + \frac{1}{t}e_\alpha$, which is 0 in $\Bbbk[\mathscr{R}_\alpha|_{(n_\alpha U_\alpha B/B)-\{n_\alpha B/B\}}] \cong \Bbbk[h_\delta, e_\gamma, t^{\pm 1}]/(e_\alpha + th_\alpha)$. Hence we can glue the preceding exact sequence with the trivial exact sequence $0 \hookrightarrow \Bbbk[h_\delta, e_\gamma, t]/(e_\alpha + th_\alpha) \twoheadrightarrow \Bbbk[h_\delta, e_\gamma, t]/(e_\alpha + th_\alpha)$ to obtain an exact sequence of sheaves

$$\mathcal{O}_{\mathscr{D}_\alpha} \hookrightarrow \mathcal{O}_{\mathscr{R}_\alpha} \twoheadrightarrow \mathcal{O}_{\mathscr{S}_\alpha}.$$

This sequence is obviously B-equivariant (the first map is non zero only over B/B, and h_α is B-invariant in our coordinate ring). This gives the first exact sequence of the lemma.

Similarly we have an exact sequence

$$\Bbbk[h_\delta, e_\gamma, t]/(h_\alpha - te_\alpha) \hookrightarrow \Bbbk[h_\delta, e_\gamma, t]/(t(h_\alpha - te_\alpha)) \twoheadrightarrow \Bbbk[h_\delta, e_\gamma, t]/(t)$$

where the first map is multiplication by t. To glue this exact sequence with the trivial one on $n_\alpha U_\alpha B/B$:

$$\Bbbk[h_\delta, e_\gamma, t]/(e_\alpha + th_\alpha) \hookrightarrow \Bbbk[h_\delta, e_\gamma, t]/(e_\alpha + th_\alpha) \twoheadrightarrow 0$$

we have to tensor $\mathcal{O}_{\mathscr{S}_\alpha}$ with the inverse image of $\mathcal{O}_{P_\alpha/B}(-\rho) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ on $P_\alpha/B \cong \mathbb{P}^1_{\Bbbk}$. We obtain the exact sequence of quasi-coherent sheaves

$$\mathcal{O}_{\mathscr{S}_\alpha}(-\rho) \hookrightarrow \mathcal{O}_{\mathscr{R}_\alpha} \twoheadrightarrow \mathcal{O}_{\mathscr{D}_\alpha}.$$

To understand the B-equivariant structure of the first morphism, we observe that to define a morphism $\mathcal{O}_{P_\alpha/B}(-\rho) \to \mathcal{O}_{P_\alpha/B}$ is equivalent to choosing a vector in $\Gamma(P_\alpha/B, \mathcal{O}_{P_\alpha/B}(\rho))$. This P_α-module has dimension two, with weights ρ and $\rho - \alpha$. The line of weight $\rho - \alpha$ is B-stable: choosing a non-zero vector in this line thus defines a morphism of B-equivariant sheaves

$$\mathcal{O}_{\mathscr{S}_\alpha}(-\rho) \otimes_{\Bbbk} \Bbbk_B(\rho - \alpha) \to \mathcal{O}_{\mathscr{R}_\alpha},$$

which yields the second exact sequence of the lemma. □

Inducing these exact sequences from B to G, we obtain

Corollary 6.2.2. *There exist exact sequences of quasi-coherent sheaves on $\mathfrak{g}^* \times (\mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B})$, where the surjections are restriction maps:*

$$\mathcal{O}_{\Delta\widetilde{\mathfrak{g}}} \hookrightarrow \mathcal{O}_{\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}} \twoheadrightarrow \mathcal{O}_{S_\alpha};$$
$$\mathcal{O}_{S_\alpha}(\rho - \alpha, -\rho) \hookrightarrow \mathcal{O}_{\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}} \twoheadrightarrow \mathcal{O}_{\Delta\widetilde{\mathfrak{g}}}.$$

Remark 6.2.3. As in Proposition 2.4.2, ρ can be replaced by any $\lambda \in \mathbb{X}$ with $\langle \lambda, \alpha^\vee \rangle = 1$.

6.3 The two actions of the braid group coincide

Assume again that $p = \mathrm{char}(\mathbb{k}) > h$. Recall the notation and results of I.1.2 and I.1.3. Let us fix some $\lambda \in \mathbb{X}$ in the alcove $\mathscr{C}_0 = \{\nu \in \mathbb{X} \otimes \mathbb{R} \mid \forall \beta \in R^+, 0 < \langle \nu + \rho, \beta^\vee \rangle < p\}$, and some $\chi \in \mathfrak{g}^*$ nilpotent. In this subsection we finally prove that the "Frobenius twisted version" of the action of B'_{aff} on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}})$ considered in Theorem 2.3.2 extends the action of B'_{aff} on $\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}_\chi^{(1)}}(\widetilde{\mathfrak{g}}^{(1)})$ coming from [BMR06, 2.1.6, 2.3.2], via equivalence $\gamma^{\mathcal{B}}_{(\lambda,\chi)}$ of (1.2.2) in chapter I. More precisely, for $b \in B'_{\mathrm{aff}}$ we denote by

$$\mathbf{J}^b : \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)}) \to \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)}), \quad \text{respectively}$$
$$\mathbf{I}^b_{(\lambda,\chi)} : \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,\chi)}(\mathcal{U}\mathfrak{g}) \to \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,\chi)}(\mathcal{U}\mathfrak{g})$$

the action of b coming from Theorem 2.3.2, respectively the action constructed in [BMR06, 2.1.4][9]. The functor $\mathbf{J}^b$ restricts to an auto-equivalence of $\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}_\chi^{(1)}}(\widetilde{\mathfrak{g}}^{(1)})$, denoted similarly. The main result of this subsection is the following:

Theorem 6.3.1. *For any $b \in B'_{\mathrm{aff}}$ there exists an isomorphism of functors from the category $\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}_\chi^{(1)}}(\widetilde{\mathfrak{g}}^{(1)})$ to itself:*

$$\mathbf{J}^b \cong (\gamma^{\mathcal{B}}_{(\lambda,\chi)})^{-1} \circ \mathbf{I}^b_{(\lambda,\chi)} \circ \gamma^{\mathcal{B}}_{(\lambda,\chi)}.$$

Proof. It is enough to consider the generators T_α (denoted by $\widetilde{s_\alpha}$ in [BMR06]) and θ_x, for $\alpha \in \Phi$ and $x \in \mathbb{X}$. First, fix some $x \in \mathbb{X}$. It is proven in [BMR06, 2.3.3] that θ_x for $x \in \mathbb{X}$ dominant acts (in the action of [BMR06]) by convolution with kernel $\Delta_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}(x)$. It follows, by construction, that this result is true for any $x \in \mathbb{X}$. Hence the two actions coincide for $b = \theta_x$.

The case of T_α is more delicate, and will occupy the rest of the proof. We fix $\alpha \in \Phi$. We will construct an isomorphism of functors

$$(\mathbf{I}^\alpha_{(\lambda,\chi)})^{-1} \circ \gamma^{\mathcal{B}}_{(\lambda,\chi)} \cong \gamma^{\mathcal{B}}_{(\lambda,\chi)} \circ F^{\mathcal{O}_{S_\alpha^{(1)}}(-\rho,\rho-\alpha)}_{\widetilde{\mathfrak{g}}^{(1)} \to \widetilde{\mathfrak{g}}^{(1)}}. \tag{6.3.2}$$

This is equivalent to the theorem for $b = T_\alpha$, due to Proposition 2.4.2. Let us choose some $\mu_\alpha \in \mathbb{X}$, on the α-wall of $\mathscr{C}_0$ (and on no other wall). We define the functor $R_\alpha := T^\lambda_{\mu_\alpha} \circ T^{\mu_\alpha}_\lambda$ (see [BMR06, 2.2.7]).

First, let us consider a single object $\mathcal{F} \in \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)})$. Now we prove that the images of $\mathcal{F}$ under the two functors in (6.3.2) are isomorphic. Later we will prove that this isomorphism comes from an isomorphism of *functors*.

Lemma 6.3.3. *There exists an isomorphism in $\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}_\chi^{(1)}}(\widetilde{\mathfrak{g}}^{(1)})$:*

$$(\mathbf{I}^\alpha_{(\lambda,\chi)})^{-1} \circ \gamma^{\mathcal{B}}_{(\lambda,\chi)}(\mathcal{F}) \cong \gamma^{\mathcal{B}}_{(\lambda,\chi)} \circ F^{\mathcal{O}_{S_\alpha^{(1)}}(-\rho,\rho-\alpha)}_{\widetilde{\mathfrak{g}}^{(1)} \to \widetilde{\mathfrak{g}}^{(1)}}(\mathcal{F}).$$

[9] This action depends on the choice of an isomorphism between the "local" extended affine braid group and B'_{aff}. We take the isomorphism associated to the choice of the element $\lambda \in W'_{\mathrm{aff}} \bullet \lambda$, as in [BMR06, 2.1.6].

Proof of Lemma 6.3.3. By definition (see [BMR06, 2.2.4, 2.3.1]), there is an exact triangle

$$(\mathbf{I}^{\alpha}_{(\lambda,\chi)})^{-1} \circ \gamma^{\mathcal{B}}_{(\lambda,\chi)}(\mathcal{F}) \to R_{\alpha} \circ \gamma^{\mathcal{B}}_{(\lambda,\chi)}(\mathcal{F}) \to \gamma^{\mathcal{B}}_{(\lambda,\chi)}(\mathcal{F}), \tag{6.3.4}$$

where the second arrow is induced by adjunction. By Propositions I.1.3.1 and 6.1.2, there exists an isomorphism $R_{\alpha} \circ \gamma^{\mathcal{B}}_{(\lambda,\chi)}(\mathcal{F}) \cong \gamma^{\mathcal{B}}_{(\lambda,\chi)} \circ F^{\mathcal{O}_{(\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_{\alpha}} \widetilde{\mathfrak{g}})^{(1)}}}_{\widetilde{\mathfrak{g}}^{(1)} \to \widetilde{\mathfrak{g}}^{(1)}}(\mathcal{F})$, and the second arrow of triangle (6.3.4) identifies with the morphism

$$\gamma^{\mathcal{B}}_{(\lambda,\chi)} \circ F^{\mathcal{O}_{(\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_{\alpha}} \mathfrak{g})^{(1)}}}_{\widetilde{\mathfrak{g}}^{(1)} \to \widetilde{\mathfrak{g}}^{(1)}}(\mathcal{F}) \to \gamma^{\mathcal{B}}_{(\lambda,\chi)}(\mathcal{F})$$

induced by the restriction map $\mathcal{O}_{(\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_{\alpha}} \mathfrak{g})^{(1)}} \to \mathcal{O}_{\Delta \widetilde{\mathfrak{g}}^{(1)}}$ (recall that the convolution with kernel $\mathcal{O}_{\Delta \widetilde{\mathfrak{g}}^{(1)}}$ is the identity). Now the result follows from the second exact sequence in Corollary 6.2.2, using basic properties of triangulated categories. $\square$

Let $q_1, q_2 : S_{\alpha}^{(1)} \to \widetilde{\mathfrak{g}}^{(1)}$ be the natural morphisms, induced by the projections $p_1, p_2 : \widetilde{\mathfrak{g}}^{(1)} \times \widetilde{\mathfrak{g}}^{(1)} \to \widetilde{\mathfrak{g}}^{(1)}$. Then, $F^{\mathcal{O}_{S_{\alpha}^{(1)}}(-\rho,\rho-\alpha)}_{\widetilde{\mathfrak{g}}^{(1)} \to \widetilde{\mathfrak{g}}^{(1)}}$ is isomorphic to the functor

$$\mathcal{F} \mapsto R(q_2)_*(L(q_1)^*\mathcal{F} \otimes_{S_{\alpha}^{(1)}} \mathcal{O}_{S_{\alpha}^{(1)}}(-\rho, \rho-\alpha))$$

(by the projection formula). We denote by $\mathcal{X}$ the completion of $\widetilde{\mathfrak{g}}^{(1)}$ along the closed subscheme $\mathcal{B}_{\chi}^{(1)}$, and by $\mathcal{Y}$ the completion of $S_{\alpha}^{(1)}$ along the closed subscheme $\mathcal{B}_{\chi}^{(1)} \times_{\mathcal{P}_{\alpha,\chi}^{(1)}} \mathcal{B}_{\chi}^{(1)}$. Then q_1 and q_2 induce morphisms of formal schemes $\widehat{q}_1, \widehat{q}_2 : \mathcal{Y} \to \mathcal{X}$. We denote by $\iota_{\mathcal{X}} : \mathcal{X} \to \widetilde{\mathfrak{g}}^{(1)}$ and $\iota_{\mathcal{Y}} : \mathcal{Y} \to S_{\alpha}^{(1)}$ the inclusion morphisms (which are flat). If $\mathcal{F}$ is in $\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})$, then $(\iota_{\mathcal{X}})^*\mathcal{F}$ is just the completion of $\mathcal{F}$ along $\mathcal{B}_{\chi}^{(1)}$ (see [Gro71, 10.8.8]), and similarly for $\mathcal{Y}$. Recall the vector bundles $\mathcal{M}^{\mathcal{B}}_{(\nu,\chi)}$ on $\mathcal{X}$ (for $\nu \in \mathbb{X}$ regular) introduced in I.1.2. Then by definition, for $\mathcal{F}$ in $\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}_{\chi}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)})$,

$$\gamma^{\mathcal{B}}_{(\nu,\chi)}(\mathcal{F}) \cong R\Gamma(\mathcal{M}^{\mathcal{B}}_{(\nu,\chi)} \otimes_{\mathcal{X}} (\iota_{\mathcal{X}})^*\mathcal{F}).$$

Let us also remark that by [BMR06, 2.2.3(c)] and the choice of vector bundles we have a functorial isomorphism

$$(\mathbf{I}^{\alpha}_{(\lambda,\chi)})^{-1} \circ \gamma^{\mathcal{B}}_{(\lambda,\chi)} \cong \gamma^{\mathcal{B}}_{(s_{\alpha}\bullet\lambda,\chi)}. \tag{6.3.5}$$

Now let $\mathcal{F} \in \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}_{\chi}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)})$. For simplicity, we write $(*)$ for the object $\gamma^{\mathcal{B}}_{(\lambda,\chi)} \circ F^{\mathcal{O}_{S_{\alpha}^{(1)}}(-\rho,\rho-\alpha)}_{\widetilde{\mathfrak{g}}^{(1)} \to \widetilde{\mathfrak{g}}^{(1)}}(\mathcal{F})$. By definition and [Gro61b, 4.1.5], we have functorial isomorphisms

$$\begin{aligned}(*) \quad &\cong \quad R\Gamma\big(\mathcal{M}^{\mathcal{B}}_{(\lambda,\chi)} \otimes_{\mathcal{X}} (\iota_{\mathcal{X}})^* R(q_2)_*(L(q_1)^*\mathcal{F} \otimes_{S_{\alpha}^{(1)}} \mathcal{O}_{S_{\alpha}^{(1)}}(-\rho,\rho-\alpha))\big) \\ &\cong \quad R\Gamma\big(\mathcal{M}^{\mathcal{B}}_{(\lambda,\chi)} \otimes_{\mathcal{X}} R(\widehat{q}_2)_*((\iota_{\mathcal{Y}})^* L(q_1)^*\mathcal{F} \otimes_{\mathcal{Y}} \mathcal{O}_{\mathcal{Y}}(-\rho,\rho-\alpha))\big).\end{aligned}$$

Now, as $q_1 \circ \iota_{\mathcal{Y}} = \iota_{\mathcal{X}} \circ \widehat{q}_1$, we deduce that

$$(*) \quad \cong \quad R\Gamma\big(\mathcal{M}^{\mathcal{B}}_{(\lambda,\chi)} \otimes_{\mathcal{X}} R(\widehat{q}_2)_*(L(\widehat{q}_1)^*(\iota_{\mathcal{X}})^*\mathcal{F} \otimes_{\mathcal{Y}} \mathcal{O}_{\mathcal{Y}}(-\rho,\rho-\alpha))\big).$$

By the projection formula applied to $\widehat{q}_2$, we have then

$$\begin{aligned}(*) \quad &\cong \quad R\Gamma \circ R(\widehat{q}_2)_*\big((\widehat{q}_2)^*\mathcal{M}^{\mathcal{B}}_{(\lambda,\chi)} \otimes_{\mathcal{Y}} L(\widehat{q}_1)^*(\iota_{\mathcal{X}})^*\mathcal{F} \otimes_{\mathcal{Y}} \mathcal{O}_{\mathcal{Y}}(-\rho,\rho-\alpha)\big) \\ &\cong \quad R\Gamma \circ R(\widehat{q}_1)_*\big((\widehat{q}_2)^*\mathcal{M}^{\mathcal{B}}_{(\lambda,\chi)} \otimes_{\mathcal{Y}} L(\widehat{q}_1)^*(\iota_{\mathcal{X}})^*\mathcal{F} \otimes_{\mathcal{Y}} \mathcal{O}_{\mathcal{Y}}(-\rho,\rho-\alpha)\big).\end{aligned}$$

Finally, the projection formula applied to $\widehat{q}_1$ gives

$$(*) \quad \cong \quad R\Gamma\big((\iota_{\mathcal{X}})^*\mathcal{F}\overset{L}{\otimes}_{\mathcal{X}} R(\widehat{q}_1)_*((\widehat{q}_2)^*\mathcal{M}^{\mathcal{B}}_{(\lambda,\chi)} \otimes_{\mathcal{Y}} \mathcal{O}_{\mathcal{Y}}(-\rho,\rho-\alpha))\big). \tag{6.3.6}$$

It follows from (6.3.5) and (6.3.6) that it is enough, to prove isomorphism (6.3.2), to construct an isomorphism

$$R(\widehat{q}_1)_*\big((\widehat{q}_2)^*\mathcal{M}^{\mathcal{B}}_{(\lambda,\chi)} \otimes_{\mathcal{Y}} \mathcal{O}_{\mathcal{Y}}(-\rho,\rho-\alpha)\big) \cong \mathcal{M}^{\mathcal{B}}_{(s_\alpha \bullet \lambda,\chi)}$$

in the derived category of coherent sheaves on $\mathcal{X}$. Let $\mathcal{I}$ be the ideal of definition of $\mathcal{B}^{(1)}_\chi$ in $\widetilde{\mathfrak{g}}^{(1)}$. By [Gro71, 10.11.3] and [Gro61b, 3.4.3], it is enough to show that for all $n \geq 1$ we have an isomorphism

$$(\mathcal{O}_{\mathcal{X}}/\mathcal{I}^n)\overset{L}{\otimes}_{\mathcal{X}} R(\widehat{q}_1)_*\big((\widehat{q}_2)^*\mathcal{M}^{\mathcal{B}}_{(\lambda,\chi)} \otimes_{\mathcal{Y}} \mathcal{O}_{\mathcal{Y}}(-\rho,\rho-\alpha)\big) \cong (\mathcal{O}_{\mathcal{X}}/\mathcal{I}^n)\overset{L}{\otimes}_{\mathcal{X}} \mathcal{M}^{\mathcal{B}}_{(s_\alpha\bullet\lambda,\chi)}.$$

Using isomorphisms (6.3.5) and (6.3.6), and the fact that $R\Gamma$ is an equivalence of categories, this isomorphism follows easily from Lemma 6.3.3 applied to $\mathcal{O}_{\mathcal{X}}/\mathcal{I}^n$. □

Remark 6.3.7. In [Bez06b], Bezrukavnikov explains the importance of this action of B'_{aff} in his plan of proof of Lusztig's conjecture concerning the representation theory of $\mathfrak{g}$. There, the definition of S_α is different from ours, but of course they are equivalent (*i.e.* they define the same subscheme of $\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$), see section 8. He also considers the action on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$ (see [Bez06b, Theorem 2.1]), without giving a proof of its existence.

7 Relation to representation theory in characteristic zero

In this section we establish a connection between our constructions in the case $\Bbbk = \mathbb{C}$ and Ginzburg's description of the equivariant K-theory of the Steinberg variety. We also relate them to Springer's action of the Weyl group on the homology of a Springer fiber.

In the whole section (except in Lemma 7.1.1) we take $\Bbbk = \mathbb{C}$.

7.1 Equivariant K-theory of the Steinberg variety

First we need a result analogous to Corollary 6.2.2, but for the action on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$. It is valid over any algebraically closed field $\Bbbk$. Consider the variety S'_α. Geometrically, it can be described as:

$$S'_\alpha = \{(X, g_1B, g_2B) \in \mathfrak{g}^* \times (\mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B}) \mid X_{|g_1\cdot\mathfrak{b}+g_2\cdot\mathfrak{b}} = 0\}.$$

It has two irreducible components. One is $\Delta\widetilde{\mathcal{N}}$, the diagonal embedding of $\widetilde{\mathcal{N}}$, and the other one is

$$Y_\alpha := \{(X, g_1B, g_2B) \in \mathfrak{g}^* \times (\mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B}) \mid X_{|g_1\cdot\mathfrak{p}_\alpha} = 0\},$$

which is a vector bundle on $\mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B}$, of rank $\dim(\mathfrak{g}/\mathfrak{b}) - 1$.

Lemma 7.1.1. *There exist exact sequences of quasi-coherent sheaves, where the surjections are restriction maps:*

$$\mathcal{O}_{\Delta\widetilde{\mathcal{N}}} \hookrightarrow \mathcal{O}_{S'_\alpha}(\rho-\alpha,-\rho) \twoheadrightarrow \mathcal{O}_{Y_\alpha}(\rho-\alpha,-\rho);$$
$$\mathcal{O}_{Y_\alpha}(\rho-\alpha,-\rho) \hookrightarrow \mathcal{O}_{S'_\alpha} \twoheadrightarrow \mathcal{O}_{\Delta\widetilde{\mathcal{N}}}.$$

Proof. The construction of the exact sequences is analogous to that in Lemma 6.2.1. Let us introduce the following subvarieties of $\mathfrak{g}^* \times (P_\alpha/B)$:

$$\mathscr{D}'_\alpha := (\mathfrak{g}/\mathfrak{b})^* \times (B/B)$$
$$\mathscr{S}'_\alpha := \{(X,gB) \in \mathfrak{g}^* \times (P_\alpha/B) \mid X_{|\mathfrak{b}+g\cdot\mathfrak{b}} = 0\}$$
$$\mathscr{Y}_\alpha := \{(X,gB) \in \mathfrak{g}^* \times (P_\alpha/B) \mid X_{|\mathfrak{p}_\alpha} = 0\}.$$

Then we have isomorphisms $\Delta\widetilde{\mathcal{N}} \cong G \times^B \mathscr{D}'_\alpha$, $S'_\alpha \cong G \times^B \mathscr{S}'_\alpha$, $Y_\alpha \cong G \times^B \mathscr{Y}_\alpha$. Let us recall the equations of the varieties $\mathscr{D}'_\alpha$, $\mathscr{S}'_\alpha$, $\mathscr{Y}_\alpha$. We use the affine covering $(P_\alpha/B) = (U_\alpha B/B) \cup (s_\alpha U_\alpha B/B)$, and the isomorphisms induced by u_α, respectively by $t \mapsto n_\alpha u_\alpha(t)$: $\mathbb{k} \cong U_\alpha B/B$, $\mathbb{k} \cong s_\alpha U_\alpha B/B$. As coordinates on $\mathfrak{g}^*$ we use the basis $\{e_\gamma, \gamma \in R, h_\beta, \beta \in \Phi\}$ of $\mathfrak{g}$. Then we can deduce from the computations in section 5 the equations defining $\mathscr{S}'_\alpha|_{U_\alpha B/B}$, $\mathscr{D}'_\alpha|_{U_\alpha B/B}$ and $\mathscr{Y}_\alpha|_{U_\alpha B/B}$ as closed suvarieties of $\mathfrak{g}^* \times \mathbb{k}$. Namely, these three varieties are defined by the equations e_γ ($\gamma \in R^-$), h_β ($\beta \in \Phi$) and, respectively, te_α, t, e_α. Hence there are exact sequences

$$\mathbb{k}[\mathscr{D}'_\alpha|_{U_\alpha B/B}] \hookrightarrow \mathbb{k}[\mathscr{S}'_\alpha|_{U_\alpha B/B}] \twoheadrightarrow \mathbb{k}[\mathscr{Y}_\alpha|_{U_\alpha B/B}],$$
$$\mathbb{k}[\mathscr{Y}_\alpha|_{U_\alpha B/B}] \hookrightarrow \mathbb{k}[\mathscr{S}'_\alpha|_{U_\alpha B/B}] \twoheadrightarrow \mathbb{k}[\mathscr{D}'_\alpha|_{U_\alpha B/B}],$$

where the first maps are respectively the multiplication by e_α and t.

Over $s_\alpha U_\alpha B/B$ we have $\mathscr{D}'_\alpha|_{s_\alpha U_\alpha B/B} = \emptyset$, $\mathscr{S}'_\alpha|_{s_\alpha U_\alpha B/B} = \mathscr{Y}_\alpha|_{s_\alpha U_\alpha B/B}$. Under the change of coordinates t is sent to $-\frac{1}{t}$, and e_α to 0. Hence there are exact sequences of quasi-coherent sheaves

$$\mathcal{O}_{\mathscr{D}'_\alpha} \hookrightarrow \mathcal{O}_{\mathscr{S}'_\alpha} \twoheadrightarrow \mathcal{O}_{\mathscr{Y}_\alpha}, \quad \mathcal{O}_{\mathscr{Y}_\alpha} \otimes_{\mathcal{O}_{P_\alpha/B}} \mathcal{O}_{P_\alpha/B}(-\rho) \hookrightarrow \mathcal{O}_{\mathscr{S}'_\alpha} \twoheadrightarrow \mathcal{O}_{\mathscr{D}'_\alpha}.$$

Concerning the B-equivariant structure, we remark that the second exact sequence was constructed just like in Lemma 6.2.1. Hence, as there we have an exact sequence of B-equivariant sheaves

$$\mathcal{O}_{\mathscr{Y}_\alpha}(-\rho) \otimes_{\mathbb{k}} \mathbb{k}_B(\rho-\alpha) \hookrightarrow \mathcal{O}_{\mathscr{S}'_\alpha} \twoheadrightarrow \mathcal{O}_{\mathscr{D}'_\alpha}.$$

Inducing from B to G we obtain the second exact sequence of the lemma. Concerning the first exact sequence, its first arrow is given by the multiplication by e_α, which has weight α for the action of B. Hence the B-equivariant exact sequence reads

$$\mathcal{O}_{\mathscr{D}'_\alpha} \otimes_{\mathbb{k}} \mathbb{k}_B(\alpha) \hookrightarrow \mathcal{O}_{\mathscr{S}'_\alpha} \twoheadrightarrow \mathcal{O}_{\mathscr{Y}_\alpha}.$$

Inducing, we obtain $\mathcal{O}_{\Delta\widetilde{\mathcal{N}}}(\alpha,0) \hookrightarrow \mathcal{O}_{S'_\alpha} \twoheadrightarrow \mathcal{O}_{Y_\alpha}$. Now $\mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(-\rho,\rho)$ is trivial on the diagonal. Hence we also have

$$\mathcal{O}_{\Delta\widetilde{\mathcal{N}}}(\alpha-\rho,\rho) \hookrightarrow \mathcal{O}_{S'_\alpha} \twoheadrightarrow \mathcal{O}_{Y_\alpha}.$$

Tensoring by the inverse image of $\mathcal{O}_{\mathcal{B}\times_{\mathcal{P}_\alpha}\mathcal{B}}(\rho-\alpha,-\rho)$, we obtain the first exact sequence of the lemma. □

Let us define a $\mathbb{C}^\times$-action on $\widetilde{\mathcal{N}}$, setting

$$t\cdot(X,gB):=(t^{-2}X,gB).$$

This action commutes with the natural action of G on $\widetilde{\mathcal{N}}$. We denote by

$$\langle 1\rangle:\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{C}^\times}(\widetilde{\mathcal{N}})\to\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{C}^\times}(\widetilde{\mathcal{N}})$$

the tensor product with the one-dimensional $\mathbb{C}^\times$-module given by $\mathrm{Id}_{\mathbb{C}^\times}$, and similarly for any variety with a $\mathbb{C}^\times$-action. Then the exact sequences of Lemma 7.1.1 have $G\times\mathbb{C}^\times$-equivariant versions

$$\mathcal{O}_{\Delta\widetilde{\mathcal{N}}}\langle 2\rangle\hookrightarrow\mathcal{O}_{S'_\alpha}(\rho-\alpha,-\rho)\twoheadrightarrow\mathcal{O}_{Y_\alpha}(\rho-\alpha,-\rho);\tag{7.1.2}$$
$$\mathcal{O}_{Y_\alpha}(\rho-\alpha,-\rho)\hookrightarrow\mathcal{O}_{S'_\alpha}\twoheadrightarrow\mathcal{O}_{\Delta\widetilde{\mathcal{N}}}.\tag{7.1.3}$$

If H is an algebraic group (over $\mathbb{C}$) acting on a variety X, we denote by $K^H(X)$ the H-equivariant K-theory of X. This is by definition the Grothendieck group of the category $\mathrm{Coh}^H(X)$ of H-equivariant coherent sheaves on X, or of its derived category $\mathcal{D}^b\mathrm{Coh}^H(X)$. We refer to [Lus98, section 6] for generalities on equivariant K-theory, and to [Bez00, section 2] and [CG97, 5.1] for the main properties of derived categories of equivariant coherent sheaves. If $\mathcal{F}$ is an object of $\mathcal{D}^b\mathrm{Coh}^H(X)$, we denote by $[\mathcal{F}]$ its image in $K^H(X)$.

Let $\mathcal{N}$ be the variety of nilpotent elements in $\mathfrak{g}^*$. We have the Springer resolution $\pi:\widetilde{\mathcal{N}}\to\mathcal{N}$. We will be interested in the Steinberg variety

$$Z:=\widetilde{\mathcal{N}}\times_{\mathcal{N}}\widetilde{\mathcal{N}},$$

and more precisely to the group $K^{G\times\mathbb{C}^\times}(Z)$. First, let us describe the ring structure on this K-group. There is a natural closed embedding $j:Z\hookrightarrow\widetilde{\mathcal{N}}^2$. Let $p_{a,b}:\widetilde{\mathcal{N}}^3\to\widetilde{\mathcal{N}}^2$ denote the projection to the a-th and b-th factors ($1\le a<b\le 3$). If $\mathcal{F}$ and $\mathcal{G}$ are in $\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{C}^\times}(Z)$, then $R(p_{1,3})_*(p_{1,2}^*(j_*\mathcal{F})\overset{L}{\otimes}_{\widetilde{\mathcal{N}}^3}p_{2,3}^*(j_*\mathcal{G}))$ is only in $\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{C}^\times}(\widetilde{\mathcal{N}}^2)$, but its cohomology is supported on Z. Hence the class $[R(p_{1,3})_*(p_{1,2}^*(j_*\mathcal{F})\overset{L}{\otimes}_{\widetilde{\mathcal{N}}^3}p_{2,3}^*(j_*\mathcal{G}))]$ is a well defined element of $K^{G\times\mathbb{C}^\times}(Z)$ (see [Bez00, 2. Lemma 3(b)], [Lus98, 6.2]). The ring structure on $K^{G\times\mathbb{C}^\times}(Z)$ is then given by the product:

$$[\mathcal{F}]\cdot[\mathcal{G}]:=[R(p_{1,3})_*(p_{1,2}^*(j_*\mathcal{F})\overset{L}{\otimes}_{\widetilde{\mathcal{N}}^3}p_{2,3}^*(j_*\mathcal{G}))].$$

Note that the unit for this product is $[\mathcal{O}_{\Delta\widetilde{\mathcal{N}}}]$.

Let v be an indeterminate, and $\mathcal{A}:=\mathbb{Z}[v,v^{-1}]$. Let $\mathcal{H}'_{\mathrm{aff}}$ be the extended affine Hecke algebra associated to R (over $\mathcal{A}$). Using the Bernstein presentation (see *e.g.* [Lus98, 1.19]) one sees that $\mathcal{H}'_{\mathrm{aff}}$ is the quotient of $\mathcal{A}[B'_{\mathrm{aff}}]$, the group algebra of B'_{aff} over $\mathcal{A}$, by the ideal generated by the relations

$$(T_\alpha+v^{-1})(T_\alpha-v)=0\tag{7.1.4}$$

for $\alpha \in \Phi$. We let $\mathcal{A}$ act on $K^{G\times\mathbb{C}^\times}(Z)$ by setting $v \cdot [\mathcal{F}] := [\mathcal{F}\langle 1\rangle]$. The varieties Y_α and S'_α are $G\times\mathbb{C}^\times$-stable subvarieties of Z, hence define natural classes $[\mathcal{O}_{Y_\alpha}]$, $[\mathcal{O}_{S'_\alpha}]$ in $K^{G\times\mathbb{C}^\times}(Z)$. If x and y are in $\mathbb{X}$, the line bundle $\mathcal{O}_Z(x,y)$ (see 2.3 for the notation) is naturally an object of $\mathrm{Coh}^{G\times\mathbb{C}^\times}(Z)$ (with trivial $\mathbb{C}^\times$-action).

As an easy consequence of our results we obtain:

Proposition 7.1.5. *The assignment*

$$\begin{cases} T_\alpha & \mapsto \quad -v^{-1}[\mathcal{O}_{Y_\alpha}(-\rho,\rho-\alpha)] - v^{-1} = -v^{-1}[\mathcal{O}_{S'_\alpha}]; \\ \theta_x & \mapsto \quad [\mathcal{O}_{\Delta\widetilde{\mathcal{N}}}(x)] \end{cases}$$

extends to a morphism of $\mathcal{A}$-algebras $\mathcal{H}'_{\mathrm{aff}} \to K^{G\times\mathbb{C}^\times}(Z)$.

Remark 7.1.6. This result is well known (see *e.g.* [Lus98, 7.25] or [CG97, 7.6.9]), and this morphism is in fact an isomorphism, as proved in [Lus98, 8.6] or [CG97, 7.6.10]. The construction of this morphism is one of the main steps of the proof of the isomorphism $\mathcal{H}'_{\mathrm{aff}} \cong K^{G\times\mathbb{C}^\times}(Z)$ (both for the proof by Ginzburg, see [Gin87] or [CG97], and for the alternate proof by Lusztig, see [Lus98]). These previous constructions are indirect, using an action on a module to prove the fact that the image of the generators satisfy the relations of $\mathcal{H}'_{\mathrm{aff}}$. Using our constructions, one can give a direct proof of the relations in $K^{G\times\mathbb{C}^\times}(Z)$ (using no K-theoretic result). Moreover, this proof gives a more concrete interpretation of the image of the generators T_α; namely, this image is a multiple of the class of $\mathcal{O}_{S'_\alpha}$.

Proof. First, the equality

$$-v^{-1}[\mathcal{O}_{Y_\alpha}(-\rho,\rho-\alpha)] - v^{-1} = -v^{-1}[\mathcal{O}_{S'_\alpha}] \tag{7.1.7}$$

follows from the exact sequence (7.1.3). We have to check that the elements $-v^{-1}[\mathcal{O}_{S'_\alpha}]$ for $\alpha \in \Phi$ and $[\mathcal{O}_{\Delta\widetilde{\mathcal{N}}}(x)]$ for $x \in \mathbb{X}$ satisfy relations 1 to 4 of Theorem 1.1.3, and the quadratic relations (7.1.4).

Relation 2 are trivial, and relations 1 and 3 follow from the results of section 5. Now the exact sequences of Lemma 2.4.5 admit the following $\mathbb{C}^\times$-equivariant versions (where the action on $\widetilde{\mathfrak{g}}$ is the natural one, extending the action on $\widetilde{\mathcal{N}}$):

$$\mathcal{O}_{V^1_\alpha}\langle 2\rangle \hookrightarrow \mathcal{O}_{V_\alpha}(\rho-\alpha,-\rho,0) \twoheadrightarrow \mathcal{O}_{V^2_\alpha}(\rho-\alpha,-\rho,0);$$
$$\mathcal{O}_{V^1_\alpha}\langle 2\rangle \hookrightarrow \mathcal{O}_{V_\alpha}(0,-\rho,\rho-\alpha) \twoheadrightarrow \mathcal{O}_{V^2_\alpha}(0,-\rho,\rho-\alpha).$$

We deduce as in section 5 that $-v^{-1}[\mathcal{O}_{S'_\alpha}]$ is invertible, and

$$(-v^{-1}[\mathcal{O}_{S'_\alpha}])^{-1} = -v^{-1}[\mathcal{O}_{S'_\alpha}(\rho-\alpha,-\rho)]. \tag{7.1.8}$$

Then relation 4 is easy to prove (as in 2.5).

Finally, for the quadratic relations, consider the exact sequence (7.1.2). It yields

$$-v^{-1}[\mathcal{O}_{S'_\alpha}(\rho-\alpha,-\rho)] = -v^{-1}[\mathcal{O}_{Y_\alpha}(\rho-\alpha,-\rho)] - v. \tag{7.1.9}$$

Using relations (7.1.7) and (7.1.8), we deduce from (7.1.9) that

$$(-v^{-1}[\mathcal{O}_{S'_\alpha}])^{-1} = (-v^{-1}[\mathcal{O}_{S'_\alpha}]) + (v^{-1}-v).$$

This is equivalent to relation (7.1.4). $\square$

It follows from these considerations that the natural action of $\mathcal{H}'_{\mathrm{aff}}$ on $K^{G\times\mathbb{C}^\times}(\widetilde{\mathcal{N}})$ (see [CG97, 7.6.6]) can be lifted to an action of B'_{aff} on the category $\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{C}^\times}(\widetilde{\mathcal{N}})$.

Remark 7.1.10. Let $\chi \in \mathfrak{g}^*$ be nilpotent, and let $\mathcal{B}_\chi$ be the corresponding Springer fiber, *i.e.* the inverse image of χ under $\widetilde{\mathfrak{g}} \to \mathfrak{g}^*$ (see I.1.1). Let M be a closed subgroup of the stabilizer of χ in $G \times \mathbb{C}^\times$, for the action defined by $(g,z)\cdot\chi = z^{-2}g\cdot\chi$. Then M stabilizes $\mathcal{B}_\chi \subset \widetilde{\mathcal{N}}$. Our constructions yield an action of B'_{aff} on $\mathcal{D}^b\mathrm{Coh}^M(\widetilde{\mathcal{N}})$, which stabilizes the full subcategory $\mathcal{D}^b\mathrm{Coh}^M_{\mathcal{B}_\chi}(\widetilde{\mathcal{N}})$ of complexes whose cohomology sheaves are supported on $\mathcal{B}_\chi$. The Grothendieck group of the category $\mathcal{D}^b\mathrm{Coh}^M_{\mathcal{B}_\chi}(\widetilde{\mathcal{N}})$ identifies with $K^M(\mathcal{B}_\chi)$. The same considerations as above show that the action of B'_{aff} induces an action of $\mathcal{H}'_{\mathrm{aff}}$ on $K^M(\mathcal{B}_\chi)$. This is the action considered in [Lus02, 3.4]. In [Lus02], Lusztig explains the importance of these modules in the construction of all the irreducible $\mathcal{H}'_{\mathrm{aff}}$-modules over $\mathbb{C}$.

7.2 Springer's representations of W

Now we consider Springer's representations of the finite Weyl group. More precisely we follow Ginzburg's approach to this question in [Gin86] (see [CG97, chapter 3] for the same arguments, in the framework of homology rather than K-theory).

As in 7.1, our constructions yield a $\mathbb{Z}$-algebra morphism

$$\mathbb{Z}[B_0] \to K(Z),$$

where $K(Z)$ is the non-equivariant K-theory of the Steinberg variety Z, and B_0 is the finite braid group (see 1.1 for the definition). The exact sequences of Lemma 7.1.1 show that for $\alpha \in \Phi$ the image of $(T_\alpha)^2$ in $K(Z)$ is 1. Hence the previous morphism gives a morphism

$$\mathbb{Z}[W] \to K(Z).$$

Following Ginzburg, we consider $K(Z)$ as the Grothendieck group of the abelian category $\mathrm{Coh}_Z(\widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}})$, and denote by $L(Z)$ the quotient by the subgroup generated by the elements $[\mathcal{F}]$ for $\mathcal{F}$ in $\mathrm{Coh}_Z(\widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}})$ such that $\dim(\mathrm{Supp}(\mathcal{F})) < \dim(Z)$. Composing the previous morphism with the natural quotient $K(Z) \to L(Z)$ we obtain a morphism

$$\mathbb{Z}[W] \to L(Z). \tag{7.2.1}$$

The following proposition follows directly from our constructions and the definition of specialization in K-theory as in [CG97, 5.3] (use the definition of S'_α as the intersection $S_\alpha \cap (\widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}})$).

Proposition 7.2.2. *The morphism* (7.2.1) *coincides with the isomorphism of* [Gin86, 5.3]*:* $\mathbb{Z}[W] \xrightarrow{\sim} L(Z)$.

This isomorphism is the main step in Ginzburg's approach to Springer's construction of the representations of W on the top homology of Springer fibers (see [CG97, 3.5-6]). Choose a nilpotent $\chi \in \mathfrak{g}^*$, and consider the Springer fiber $\mathcal{B}_\chi$ (see 7.1). As noted above, the B'_{aff}-action on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}})$ induces an action of B_0 on $K(\mathcal{B}_\chi)$ (this is the case $M=1$ in Remark 7.1.10), which factorizes through the finite Weyl group W (for the same reason as above). This in turn induces an action of W on $L(\mathcal{B}_\chi)$, the quotient of $K(\mathcal{B}_\chi)$ defined as above for $L(Z)$. By Grothendieck-Riemann-Roch, we have an isomorphism $L(\mathcal{B}_\chi)\otimes_{\mathbb{Z}}\mathbb{Q}\cong H^{\mathrm{top}}(\mathcal{B}_\chi,\mathbb{Q})$. Via this isomorphism, the action of W gives Springer's action on $H^{\mathrm{top}}(\mathcal{B}_\chi,\mathbb{Q})$ (by Proposition 7.2.2 and [CG97, 3.5-6]).

8 Alternate proof of the finite braid relations

In this section we assume that p is very good for G. We give a different proof of the finite braid relations (relations 1 of Theorem 1.1.3), which is valid for any group G (and p very good), and avoids case-by-case considerations. It is a joint work with Roman Bezrukavnikov. This will complete the proof of Theorem 2.3.2.

As above (see 2.3), if $\lambda\in\mathbb{X}$, and if $X\to\mathcal{B}$ is a variety over $\mathcal{B}$, we denote by $\mathcal{O}_X(\lambda)$ the inverse image of $\mathcal{O}_{\mathcal{B}}(\lambda)$. More generally, if P is a parabolic subgroup of G and V is any finite dimensional P-module, there exists a natural vector bundle $\mathcal{L}_{G/P}(V)$ on G/P associated to V (see [Jan03, I.5.8]). If $X\to G/P$ is a variety over G/P, we denote by $\mathcal{L}_X(V)$ the inverse image of $\mathcal{L}_{G/P}(V)$.

If $\mathscr{B}$ is a triangulated category, and $\mathscr{A}\subset\mathscr{B}$ is a full triangulated subcategory, for $M,N\in\mathscr{B}$ we write $M\cong N$ mod $\mathscr{A}$ if the images of M and N in the quotient category $\mathscr{B}/\mathscr{A}$ are isomorphic.

8.1 Line bundles on $\widetilde{\mathfrak{g}}$

The methods of this section come from [Bez06a]. We use the same notation as above for the convolution functors (see 2.1).

As p is very good, there exists a G-equivariant isomorphism $\mathfrak{g}\cong\mathfrak{g}^*$. Under this isomorphism, $\widetilde{\mathfrak{g}}$ identifies with the induced variety $G\times^B\mathfrak{b}$.

Now consider the projective morphism $\widetilde{\mathfrak{g}}\to\mathfrak{g}^*$. Let $\mathfrak{g}_{\mathrm{reg}}\subset\mathfrak{g}$ denote the open set of regular elements, and $\mathfrak{g}^*_{\mathrm{reg}}\subset\mathfrak{g}^*$ the image of $\mathfrak{g}_{\mathrm{reg}}$ under the isomorphism $\mathfrak{g}\cong\mathfrak{g}^*$. Let $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$ be the inverse image of $\mathfrak{g}^*_{\mathrm{reg}}$ under the morphism $\widetilde{\mathfrak{g}}\to\mathfrak{g}^*$. There is a natural action of W on $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$ (see *e.g.* [Jan04][10]). Moreover, for $\alpha\in\Phi$, S_α is the closure of the graph of the action of s_α (indeed, S_α contains this closure, and both varieties are irreducible of the same dimension).

Let us consider the category $\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathfrak{g}})$, where G acts on $\widetilde{\mathfrak{g}}$ via the natural action and $\mathbb{G}_{\mathbf{m}}\cong\Bbbk^\times$ acts by dilatation along the fibers: for $t\in\Bbbk^\times$ and $(X,gB)\in\widetilde{\mathfrak{g}}$ we put

$$t\cdot(X,gB)=(t^2X,gB).$$

[10] In fact, in [Jan04] the author proves that there is a W-action on $\widetilde{\mathfrak{g}}_{\mathrm{rs}}$, the inverse image of regular *semi-simple* elements in $\mathfrak{g}^*$. We could not find a reference for the construction of a W-action on the whole of $\widetilde{\mathfrak{g}}_{\mathrm{reg}}$. However, we will use only very easy facts on this action, which can be checked "by hand". For instance, one can use the previous description of S_α to *define* the action of s_α.

For $\lambda \in \mathbb{X}$, the linde bundle $\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)$ is an object of $\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{G}_\mathbf{m}}(\widetilde{\mathfrak{g}})$ (with trivial $\mathbb{G}_\mathbf{m}$-action). As in section 7, we denote by

$$\langle 1\rangle : \mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{G}_\mathbf{m}}(\widetilde{\mathfrak{g}}) \to \mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{G}_\mathbf{m}}(\widetilde{\mathfrak{g}})$$

the shift functor, *i.e.* the tensor product with the 1-dimensional $\Bbbk^\times$-module given by $\mathrm{Id}_{\Bbbk^\times}$. We denote by $\langle j\rangle$ the j-th power of $\langle 1\rangle$, for $j \in \mathbb{Z}$.

If A is any subset of $\mathbb{X}$, we denote by $\mathcal{D}_A$ the smallest strictly full triangulated subcategory of $\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{G}_\mathbf{m}}(\widetilde{\mathfrak{g}})$ containing the line bundles $\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)$ for $\lambda \in A$ and stable under the functor $\langle 1\rangle$.

We denote by $\mathrm{conv}(\lambda)$ the intersection of $\mathbb{X}$ with the convex hull of $W \cdot \lambda$, and by $\mathrm{conv}^0(\lambda)$ the complement of $W \cdot \lambda$ in $\mathrm{conv}(\lambda)$.

Lemma 8.1.1. *Let $\alpha \in \Phi$.*

(i) *For any $\lambda \in \mathbb{X}$, the functors $F^{\mathcal{O}_{S_\alpha}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}}$, $F^{\mathcal{O}_{S_\alpha}(-\rho,\rho-\alpha)}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}}$ preserve the subcategory $\mathcal{D}_{\mathrm{conv}(\lambda)}$.*

(ii) *Let $\lambda \in \mathbb{X}$ such that $\langle \lambda, \alpha^\vee\rangle \leq 0$. Then*

$$F^{\mathcal{O}_{S_\alpha}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}}(\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)) \cong \mathcal{O}_{\widetilde{\mathfrak{g}}}(s_\alpha\lambda)\ \langle -2\rangle \mod \mathcal{D}_{\mathrm{conv}^0(\lambda)}.$$

(iii) *Let $\lambda \in \mathbb{X}$ such that $\langle \lambda, \alpha^\vee\rangle \geq 0$. Then*

$$F^{\mathcal{O}_{S_\alpha}(-\rho,\rho-\alpha)}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}}(\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)) \cong \mathcal{O}_{\widetilde{\mathfrak{g}}}(s_\alpha\lambda) \mod \mathcal{D}_{\mathrm{conv}^0(\lambda)}.$$

Proof. As above, let P_α be the minimal standard parabolic subgroup of G associated to $\{\alpha\}$, and let $\mathcal{P}_\alpha := G/P_\alpha$ be the corresponding partial flag variety. We have defined in I.1.1 the variety $\widetilde{\mathfrak{g}}_\alpha$. It is endowed with a natural $G\times\mathbb{G}_\mathbf{m}$-action, such that the morphism $\widetilde{\pi}_\alpha : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}_\alpha$ is $G\times\mathbb{G}_\mathbf{m}$-equivariant. By Proposition 6.1.2 and Corollary 6.2.2, for any $\mathcal{F}$ in $\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{G}_\mathbf{m}}(\widetilde{\mathfrak{g}})$ there exist distinguished triangles

$$\mathcal{F}\langle -2\rangle \ \to\ L(\widetilde{\pi}_\alpha)^* \circ R(\widetilde{\pi}_\alpha)_*\mathcal{F} \ \to\ F^{\mathcal{O}_{S_\alpha}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}}(\mathcal{F}); \tag{8.1.2}$$

$$F^{\mathcal{O}_{S_\alpha}(-\rho,\rho-\alpha)}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}}(\mathcal{F}) \ \to\ L(\widetilde{\pi}_\alpha)^* \circ R(\widetilde{\pi}_\alpha)_*\mathcal{F} \ \to\ \mathcal{F}. \tag{8.1.3}$$

Let $i : \widetilde{\mathfrak{g}} \hookrightarrow \widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B}$ be the natural inclusion. There exists an exact sequence

$$\mathcal{O}_{\widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B}}(-\alpha)\langle -2\rangle \ \hookrightarrow\ \mathcal{O}_{\widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B}} \ \twoheadrightarrow\ i_*\mathcal{O}_{\widetilde{\mathfrak{g}}} \tag{8.1.4}$$

(because $\mathfrak{b} \subset \mathrm{Lie}(P_\alpha)$ is defined by one equation, of weight $(-\alpha,-2)$ for $G\times\mathbb{G}_\mathbf{m}$). Let also $p : \widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B} \to \widetilde{\mathfrak{g}}_\alpha$ be the projection. Then $\widetilde{\pi}_\alpha = p \circ i$.

Using triangles (8.1.2) and (8.1.3), to prove (i) it is sufficient to prove that for any $\lambda \in \mathbb{X}$, $L(\widetilde{\pi}_\alpha)^* \circ R(\widetilde{\pi}_\alpha)_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)$ is in $\mathcal{D}_{\mathrm{conv}(\lambda)}$. The case $\langle \lambda, \alpha^\vee\rangle = 0$ is trivial. First, assume that $\langle \lambda, \alpha^\vee\rangle > 0$. Tensoring (8.1.4) by $\mathcal{O}_{\widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B}}(\lambda)$ we obtain an exact sequence

$$\mathcal{O}_{\widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B}}(\lambda-\alpha)\langle -2\rangle \ \hookrightarrow\ \mathcal{O}_{\widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B}}(\lambda) \ \twoheadrightarrow\ i_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda).$$

Then, applying the functor Rp_* and using [Jan03, I.5.19, II.5.2] we obtain a distinguished triangle

$$\mathcal{L}_{\widetilde{\mathfrak{g}}_\alpha}(\mathrm{Ind}_B^{P_\alpha}(\lambda-\alpha))\langle -2\rangle \ \to\ \mathcal{L}_{\widetilde{\mathfrak{g}}_\alpha}(\mathrm{Ind}_B^{P_\alpha}(\lambda)) \ \to\ R(\widetilde{\pi}_\alpha)_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda).$$

Applying the functor $L(\widetilde{\pi}_\alpha)^*$ we obtain a triangle

$$\mathcal{L}_{\widetilde{\mathfrak{g}}}(\mathrm{Ind}_B^{P_\alpha}(\lambda-\alpha))\langle -2\rangle \ \to\ \mathcal{L}_{\widetilde{\mathfrak{g}}}(\mathrm{Ind}_B^{P_\alpha}(\lambda)) \ \to\ L(\widetilde{\pi}_\alpha)^* \circ R(\widetilde{\pi}_\alpha)_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda).$$

Now it is well known (see again [Jan03, II.5.2]) that the P_α-module $\mathrm{Ind}_B^{P_\alpha}(\lambda)$ has weights λ, $\lambda-\alpha$, $\cdots$, $s_\alpha\lambda$. Hence $\mathcal{L}_{\widetilde{\mathfrak{g}}}(\mathrm{Ind}_B^{P_\alpha}(\lambda))$ has a filtration with subquotients $\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)$, $\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda-\alpha)$, $\cdots$, $\mathcal{O}_{\widetilde{\mathfrak{g}}}(s_\alpha\lambda)$. Similarly, $\mathcal{L}_{\widetilde{\mathfrak{g}}}(\mathrm{Ind}_B^{P_\alpha}(\lambda-\alpha))$ has a filtration with subquotients $\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda-\alpha)$, $\cdots$, $\mathcal{O}_{\widetilde{\mathfrak{g}}}(s_\alpha\lambda+\alpha)$. This proves (i) in this case, and also (iii).

Now assume $\langle\lambda,\alpha^\vee\rangle<0$. Using similar arguments, there exists a distinguished triangle

$$L(\widetilde{\pi}_\alpha)^* \circ R(\widetilde{\pi}_\alpha)_*\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda) \ \to\ \mathcal{L}_{\widetilde{\mathfrak{g}}}(R^1\mathrm{Ind}_B^{P_\alpha}(\lambda-\alpha))\langle -2\rangle \ \to\ \mathcal{L}_{\widetilde{\mathfrak{g}}}(R^1\mathrm{Ind}_B^{P_\alpha}(\lambda)).$$

Moreover, $\mathcal{L}_{\widetilde{\mathfrak{g}}}(R^1\mathrm{Ind}_B^{P_\alpha}(\lambda))$ has a filtration with subquotients $\mathcal{O}_{\widetilde{\mathfrak{g}}}(s_\alpha\lambda-\alpha)$, $\cdots$, $\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda+\alpha)$, and $\mathcal{L}_{\widetilde{\mathfrak{g}}}(R^1\mathrm{Ind}_B^{P_\alpha}(\lambda-\alpha))$ has a filtration with subquotients $\mathcal{O}_{\widetilde{\mathfrak{g}}}(s_\alpha\lambda)$, $\cdots$, $\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)$. As above, this proves (i) in this case, and (ii). ☐

Lemma 8.1.5. *Let $\lambda,\mu\in\mathbb{X}$.*

We have $\mathrm{Ext}^*_{\mathcal{D}^b\mathrm{Coh}^G(\widetilde{\mathfrak{g}})}(\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda),\mathcal{O}_{\widetilde{\mathfrak{g}}}(\mu))=0$ unless $\lambda-\mu\in\mathbb{Z}_{\geq 0}R^+$.

Similarly, for any $i\in\mathbb{Z}$ we have $\mathrm{Ext}^*_{\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{G}_\mathbf{m}}(\widetilde{\mathfrak{g}})}(\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda),\mathcal{O}_{\widetilde{\mathfrak{g}}}(\mu)\langle i\rangle)=0$ unless $\lambda-\mu\in\mathbb{Z}_{\geq 0}R^+$.

Proof. This proof is generalization of that of [Bez06a, Lemma 5].

We give a proof only in the first case. Recall that $\mathcal{D}^b\mathrm{Coh}^G(\widetilde{\mathfrak{g}})$ is equivalent to the full subcategory of $\mathcal{D}^b\mathrm{QCoh}^G(\widetilde{\mathfrak{g}})$ whose objects have coherent cohomology (see [Bez00, Corollary 1]). Hence we can replace $\mathcal{D}^b\mathrm{Coh}^G(\widetilde{\mathfrak{g}})$ by $\mathcal{D}^b\mathrm{QCoh}^G(\widetilde{\mathfrak{g}})$ in the statement. Moreover, for any $i\in\mathbb{Z}$ there is a natural isomorphism

$$\mathrm{Ext}^i_{\mathcal{D}^b\mathrm{QCoh}^G(\widetilde{\mathfrak{g}})}(\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda),\mathcal{O}_{\widetilde{\mathfrak{g}}}(\mu)) \cong H^i\big(R(\Gamma^G)(\mathcal{O}_{\widetilde{\mathfrak{g}}}(\mu-\lambda))\big), \tag{8.1.6}$$

where Γ^G denotes the functor which sends a G-equivariant quasi-coherent sheaf $\mathcal{F}$ to the G-invariants in its global sections.

Recall also that, via the isomorphism $\widetilde{\mathfrak{g}}\cong G\times^B\mathfrak{b}$, the restriction functor $\mathcal{F}\mapsto\mathcal{F}|_{\{1\}\times\mathfrak{b}}$ induces an equivalence of categories

$$\mathrm{QCoh}^G(\widetilde{\mathfrak{g}}) \xrightarrow{\sim} \mathrm{QCoh}^B(\mathfrak{b})$$

(see *e.g.* [Bri03, section 2]). Moreover, the following diagram commutes, where Γ^B is defined as Γ^G above:

$$\begin{tikzcd}
\mathrm{QCoh}^G(\widetilde{\mathfrak{g}}) \arrow[d, "\sim"] \arrow[dr, "\Gamma^G"] & \\
\mathrm{QCoh}^B(\mathfrak{b}) \arrow[r, "\Gamma^B"] & \mathrm{Vect}(\Bbbk).
\end{tikzcd}$$

It follows, using isomorphism (8.1.6), that for any $i \in \mathbb{Z}$ we have

$$\mathrm{Ext}^i_{\mathcal{D}^b\mathrm{QCoh}^G(\widetilde{\mathfrak{g}})}(\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda), \mathcal{O}_{\widetilde{\mathfrak{g}}}(\mu)) \cong H^i\big(R(\Gamma^B)(\mathcal{O}_{\mathfrak{b}} \otimes_{\Bbbk} \Bbbk_B(\mu - \lambda))\big). \tag{8.1.7}$$

The functor Γ^B is the composition of the functor

$$\Gamma(\mathfrak{b}, -) : \mathrm{QCoh}^B(\mathfrak{b}) \xrightarrow{\sim} \mathrm{Mod}^B(\mathrm{S}(\mathfrak{b}^*)),$$

which is an equivalence of categories because $\mathfrak{b}$ is affine, and the B-fixed points functor $I^B : \mathrm{Mod}^B(\mathrm{S}(\mathfrak{b}^*)) \to \mathrm{Vect}(\Bbbk)$. Hence, using isomorphism (8.1.7) we deduce that for any $i \in \mathbb{Z}$ we have

$$\mathrm{Ext}^i_{\mathcal{D}^b\mathrm{QCoh}^G(\widetilde{\mathfrak{g}})}(\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda), \mathcal{O}_{\widetilde{\mathfrak{g}}}(\mu)) \cong H^i\big(R(I^B)(\mathrm{S}(\mathfrak{b}^*) \otimes_{\Bbbk} \Bbbk_B(\mu - \lambda))\big). \tag{8.1.8}$$

Now I^B is the composition of the forgetful functor $\mathrm{For} : \mathrm{Mod}^B(\mathrm{S}(\mathfrak{b}^*)) \to \mathrm{Rep}(B)$ and the B-fixed points functor $J^B : \mathrm{Rep}(B) \to \mathrm{Vect}(\Bbbk)$. Of course the functor For is exact, and in the category $\mathrm{Mod}^B(\mathrm{S}(\mathfrak{b}^*))$ there are enough objects of the form $\mathrm{Ind}^B_{\{1\}}(M) \cong M \otimes_{\Bbbk} \Bbbk[B]$, for M a $\mathrm{S}(\mathfrak{b}^*)$-module, whose images under For are acyclic for the functor J^B. Hence for any $i \in \mathbb{Z}$ we have

$$\mathrm{Ext}^i_{\mathcal{D}^b\mathrm{QCoh}^G(\widetilde{\mathfrak{g}})}(\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda), \mathcal{O}_{\widetilde{\mathfrak{g}}}(\mu)) \cong H^i\big(R(J^B)(\mathrm{S}(\mathfrak{b}^*) \otimes_{\Bbbk} \Bbbk_B(\mu - \lambda))\big), \tag{8.1.9}$$

where for simplicity we have omitted the functor For.

Finally, as $B \cong T \ltimes U$, the functor J^B is the composition of the U-fixed points functor J^U, followed by the T-fixed points functor J^T (which is exact). Hence $RJ^B \cong J^T \circ RJ^U$, and we have to prove that

$$J^T\big(R(J^U)(\mathrm{S}(\mathfrak{b}^*) \otimes_{\Bbbk} \Bbbk_B(\nu))\big) = 0 \tag{8.1.10}$$

unless ν is a sum of negative roots. But $R(J^U)(\mathrm{S}(\mathfrak{b}^*) \otimes_{\Bbbk} \Bbbk_B(\nu))$ can be computed by the Hochschild complex $C(U,\ \mathrm{S}(\mathfrak{b}^*) \otimes_{\Bbbk} \Bbbk_B(\nu))$ (see [Jan03, I.4.16]). And the T-weights of this complex are all in $\mathbb{Z}_{\geq 0}R^+$ (because all weights of $\mathrm{S}(\mathfrak{b}^*)$ and of $\Bbbk[U]$ are in $\mathbb{Z}_{\geq 0}R^+$). Then (8.1.10) easily follows. □

Lemma 8.1.11. *Let $\lambda \in \mathbb{X}$, such that $\lambda - \rho$ is dominant. Then $\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)$ is an ample line bundle on $\widetilde{\mathfrak{g}}$.*

Proof. By definition $\widetilde{\mathfrak{g}}$ is a closed subscheme of $\mathfrak{g}^* \times \mathcal{B}$. Hence it is sufficient to prove that $\mathcal{O}_{\mathfrak{g}^* \times \mathcal{B}}(\lambda)$ is ample. But $\mathcal{O}_{\mathcal{B}}(\lambda)$ is very ample on $\mathcal{B}$ (see [Jan03, II.8.5]). Hence $\mathcal{O}_{\mathfrak{g}^* \times \mathcal{B}}(\lambda)$ is also very ample. □

8.2 Braid relations

Proposition 8.2.1. *Let $\alpha, \beta \in \Phi$. For any $\lambda \in \mathbb{X}^+$ we have an isomorphism*

$$F^{\mathcal{O}_{S_\alpha}(-\rho,\rho-\alpha)}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}} \circ F^{\mathcal{O}_{S_\beta}(-\rho,\rho-\beta)}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}} \circ \cdots \big(\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)\big) \ \cong\ F^{\mathcal{O}_{S_\beta}(-\rho,\rho-\beta)}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}} \circ F^{\mathcal{O}_{S_\alpha}(-\rho,\rho-\alpha)}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}} \circ \cdots \big(\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)\big),$$

where the number of functors appearing on each side is the order of $s_\alpha s_\beta$ in W.

Proof. To fix notations, let us assume that α and β generate a sub-system of type $\mathbf{A}_2$ (the proof is similar in the other cases). By Proposition 2.4.2 we have an isomorphism of functors

$$\big(F^{\mathcal{O}_{S_\alpha}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}}\big)^{-1} \cong F^{\mathcal{O}_{S_\alpha}(-\rho,\rho-\alpha)}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}}\langle 2\rangle,$$

and similarly for β. Hence proving the proposition is equivalent to proving that

$$E_\lambda := F^{\mathcal{O}_{S_\alpha}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}} \circ F^{\mathcal{O}_{S_\beta}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}} \circ F^{\mathcal{O}_{S_\alpha}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}} \circ F^{\mathcal{O}_{S_\beta}(-\rho,\rho-\beta)}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}} \circ F^{\mathcal{O}_{S_\alpha}(-\rho,\rho-\alpha)}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}} \circ F^{\mathcal{O}_{S_\beta}(-\rho,\rho-\beta)}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}}(\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)) \quad (8.2.2)$$

is isomorphic to $\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)\langle -6\rangle$. First, it follows from Lemma 8.1.1 that

$$E_\lambda \cong \mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)\langle -6\rangle \text{ mod } \mathcal{D}_{\mathrm{conv}^0(\lambda)}. \quad (8.2.3)$$

For any full subcategory $\mathcal{A}$ of a category $\mathcal{B}$, we denote by $(\mathcal{A}^\perp)_{\mathcal{B}}$ the full subcategory of $\mathcal{B}$ with objects the M such that $\mathrm{Hom}_{\mathcal{B}}(A,M)=0$ for any A in $\mathcal{A}$. By Lemma 8.1.5, $\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)$ is in $(\mathcal{D}^\perp_{\mathrm{conv}^0(\lambda)})_{\mathcal{D}_{\mathrm{conv}^0(\lambda)\cup\{\lambda\}}}$. Hence, as all the functors involved preserve the subcategory $(\mathcal{D}^\perp_{\mathrm{conv}^0(\lambda)})_{\mathcal{D}_{\mathrm{conv}(\lambda)}}$ (because their inverse preserves $\mathcal{D}_{\mathrm{conv}^0(\lambda)}$ by Lemma 8.1.1), also E_λ is in $(\mathcal{D}^\perp_{\mathrm{conv}^0(\lambda)})_{\mathcal{D}_{\mathrm{conv}^0(\lambda)\cup\{\lambda\}}}$. (Observe that E_λ is in $\mathcal{D}_{\mathrm{conv}^0(\lambda)\cup\{\lambda\}}$ by (8.2.3).) Now it follows easily from [BK90, 1.5, 1.6] that the projection

$$(\mathcal{D}^\perp_{\mathrm{conv}^0(\lambda)})_{\mathcal{D}_{\mathrm{conv}^0(\lambda)\cup\{\lambda\}}} \to \mathcal{D}_{\mathrm{conv}^0(\lambda)\cup\{\lambda\}}/\mathcal{D}_{\mathrm{conv}^0(\lambda)}$$

is an equivalence of categories. Using again (8.2.3), we deduce that $E_\lambda \cong \mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)\langle -6\rangle$ in $\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{G}_\mathbf{m}}(\widetilde{\mathfrak{g}})$, as claimed. □

Before the next corollary we introduce some notation. We denote by $\mathbb{X}^+ \subset \mathbb{X}$ the dominant weights. If λ is a dominant weight, we write that a property is true for $\lambda \gg 0$ if there exists a positive integer N such that the property is true for any weight λ such that $\langle\lambda,\alpha^\vee\rangle \geq N$ for any positive root α.

Corollary 8.2.4. *The kernels $\mathcal{O}_{S_\alpha}$, $\alpha\in\Phi$, satisfy the finite braid relations in the category $\mathcal{D}^b\mathrm{Coh}_{\mathrm{prop}}(\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}})$. In other words, for $\alpha,\beta\in\Phi$ there exists an isomorphism*

$$\mathcal{O}_{S_\alpha} * \mathcal{O}_{S_\beta} * \cdots \;\cong\; \mathcal{O}_{S_\beta} * \mathcal{O}_{S_\alpha} * \cdots,$$

where the number of terms on each side is the order of $s_\alpha s_\beta$ in W.

Proof. To fix notations, let us assume that α and β generate a root system of type $\mathbf{A}_2$ (the other cases are similar). The kernels $\mathcal{O}_{S_\alpha}$, $\mathcal{O}_{S_\beta}$ are invertible (see Proposition 2.4.2), hence we only have to prove that

$$\mathcal{O}_{S_\alpha} * \mathcal{O}_{S_\beta} * \mathcal{O}_{S_\alpha} * (\mathcal{O}_{S_\beta})^{-1} * (\mathcal{O}_{S_\alpha})^{-1} * (\mathcal{O}_{S_\beta})^{-1} \cong \mathcal{O}_{\Delta\widetilde{\mathfrak{g}}}.$$

To simplify notations, let us denote by $\mathcal{K}_{\alpha,\beta}$ the object on the left hand side of this equation. To prove the isomorphism it is sufficient, using Lemma 8.1.11, to prove that for $\lambda,\mu \gg 0$

we have $R^{\neq 0}\Gamma(\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}},\ \mathcal{K}_{\alpha,\beta}(\lambda,\mu)) = 0$ (this implies that $\mathcal{K}_{\alpha,\beta}$ is concentrated in degree 0, *i.e.* is a sheaf), and that there exist isomorphisms

$$\Gamma(\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}},\ \mathcal{K}_{\alpha,\beta}(\lambda,\mu)) \cong \Gamma(\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}},\ \mathcal{O}_{\Delta\widetilde{\mathfrak{g}}}(\lambda,\mu)),$$

compatible with the natural action of $\bigoplus_{\eta,\nu \in \mathbb{X}^+} \Gamma(\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}},\ \mathcal{O}_{\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}}}(\eta,\nu))$.

The object $\mathcal{K}_{\alpha,\beta}$ is the kernel associated to the functor

$$F_{\alpha,\beta} := F^{\mathcal{O}_{S_\alpha}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}} \circ F^{\mathcal{O}_{S_\beta}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}} \circ F^{\mathcal{O}_{S_\alpha}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}} \circ (F^{\mathcal{O}_{S_\beta}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}})^{-1} \circ (F^{\mathcal{O}_{S_\alpha}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}})^{-1} \circ (F^{\mathcal{O}_{S_\beta}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}})^{-1}.$$

We have seen in Proposition 8.2.1 that $F_{\alpha,\beta}$ fixes any line bundle $\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)$ with $\lambda \in \mathbb{X}^+$. Moreover, for any λ, μ we have, by the projection formula, $R\Gamma(\widetilde{\mathfrak{g}},\ F^{\mathcal{K}_{\alpha,\beta}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}}(\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)) \otimes_{\mathcal{O}_{\widetilde{\mathfrak{g}}}} \mathcal{O}_{\widetilde{\mathfrak{g}}}(\mu)) \cong R\Gamma(\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}},\ \mathcal{K}_{\alpha,\beta}(\lambda,\mu))$. It follows, using [Har77, III.5.2] and the fact that the morphism $\widetilde{\mathfrak{g}} \to \mathfrak{g}^*$ is projective, that for $\lambda, \mu \gg 0$ we have $R^{\neq 0}(\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}},\ \mathcal{K}_{\alpha,\beta}(\lambda,\mu)) = 0$ and, moreover, there is an isomorphism $\Gamma(\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}},\ \mathcal{K}_{\alpha,\beta}(\lambda,\mu)) \cong \Gamma(\widetilde{\mathfrak{g}},\ \mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda+\mu)) \cong \Gamma(\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}},\ \mathcal{O}_{\Delta\widetilde{\mathfrak{g}}}(\lambda,\mu))$.

It remains to show that these isomorphisms can be chosen in a way compatible with the action of $\bigoplus_{\eta,\nu\in\mathbb{X}^+} \Gamma(\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}},\ \mathcal{O}_{\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}}}(\eta,\nu))$. To prove this, observe that the isomorphism $F^{\mathcal{K}_{\alpha,\beta}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}}(\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)) \cong \mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)$ in $\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{G}_\mathbf{m}}(\widetilde{\mathfrak{g}})$ proved in Proposition 8.2.1 is unique up to a scalar, because

$$\mathrm{Hom}_{\mathcal{D}^b\mathrm{Coh}^{G\times\mathbb{G}_\mathbf{m}}(\widetilde{\mathfrak{g}})}(\mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda),\ \mathcal{O}_{\widetilde{\mathfrak{g}}}(\lambda)) \cong \Gamma(\widetilde{\mathfrak{g}},\ \mathcal{O}_{\widetilde{\mathfrak{g}}})^{G\times\mathbb{G}_\mathbf{m}} = \Bbbk.$$

Let us show that there is a canonical choice of this scalar. Let $j : \widetilde{\mathfrak{g}}_{\mathrm{reg}} \hookrightarrow \widetilde{\mathfrak{g}}$ be the open embedding, and let S^0_α be the restriction of S_α to $\widetilde{\mathfrak{g}}_{\mathrm{reg}} \times \widetilde{\mathfrak{g}}_{\mathrm{reg}}$. Then there is a canonical isomorphism $j^* \circ F^{\mathcal{O}_{S_\alpha}}_{\widetilde{\mathfrak{g}}\to\widetilde{\mathfrak{g}}} \cong F^{\mathcal{O}_{S^0_\alpha}}_{\widetilde{\mathfrak{g}}_{\mathrm{reg}}\to\widetilde{\mathfrak{g}}_{\mathrm{reg}}} \circ j^*$, and similarly for β and the inverse functors. Moreover it is clear that there is a canonical isomorphism of endofunctors of $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}_{\mathrm{reg}})$:

$$F^{\mathcal{O}_{S^0_\alpha}}_{\widetilde{\mathfrak{g}}_{\mathrm{reg}}\to\widetilde{\mathfrak{g}}_{\mathrm{reg}}} \circ F^{\mathcal{O}_{S^0_\beta}}_{\widetilde{\mathfrak{g}}_{\mathrm{reg}}\to\widetilde{\mathfrak{g}}_{\mathrm{reg}}} \circ F^{\mathcal{O}_{S^0_\alpha}}_{\widetilde{\mathfrak{g}}_{\mathrm{reg}}\to\widetilde{\mathfrak{g}}_{\mathrm{reg}}} \circ (F^{\mathcal{O}_{S^0_\beta}}_{\widetilde{\mathfrak{g}}_{\mathrm{reg}}\to\widetilde{\mathfrak{g}}_{\mathrm{reg}}})^{-1} \circ (F^{\mathcal{O}_{S^0_\alpha}}_{\widetilde{\mathfrak{g}}_{\mathrm{reg}}\to\widetilde{\mathfrak{g}}_{\mathrm{reg}}})^{-1} \circ (F^{\mathcal{O}_{S^0_\beta}}_{\widetilde{\mathfrak{g}}_{\mathrm{reg}}\to\widetilde{\mathfrak{g}}_{\mathrm{reg}}})^{-1} \cong \mathrm{Id}.$$

We choose the scalars above so that the corresponding isomorphism is compatible with this canonical isomorphism. (Note that $\widetilde{\mathfrak{g}}_{\mathrm{reg}} \subset \widetilde{\mathfrak{g}}$ is an open subscheme of codimension 2; hence the restriction induces an isomorphism $\Gamma(\widetilde{\mathfrak{g}}, \mathcal{O}_{\widetilde{\mathfrak{g}}}) \xrightarrow{\sim} \Gamma(\widetilde{\mathfrak{g}}_{\mathrm{reg}}, \mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathrm{reg}}})$.) Then the compatibility of the actions of the algebra $\bigoplus_{\eta,\nu\in\mathbb{X}^+} \Gamma(\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}},\ \mathcal{O}_{\widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}}}(\eta,\nu))$ is clear. $\square$

It follows from this corollary that Theorem 2.3.2 is valid for any group G and very good characteristic p. In particular, the results of section 5 are true in this generality, and those of sections 6 and 7 are true in complete generality.

Chapter III

Koszul duality and $\mathcal{U}\mathfrak{g}$-modules

This chapter countains our main constructions. Using geometric methods we build, for any regular $\lambda \in \mathbb{X}$, a "Koszul-type" duality between the categories $\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^\lambda)$ and $\mathcal{D}^b\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$, and show that it sends simples to projectives. We also study a "parabolic analogue" of these constructions, and apply our results to Koszulity of blocks of the restricted enveloping algebra.

This chapter was prepublished in [Ric08b].

Introduction

0.1

Since [BGS96], Koszul duality has proved to be a very useful and powerful tool in Lie theory. In [BGS96], Beilinson, Ginzburg and Soergel prove that every block of the category $\mathcal{O}$ of a complex semi-simple Lie algebra is governed by a Koszul ring, whose dual ring governs another subcategory of the category $\mathcal{O}$. In this chapter we obtain, using completely different methods, counterparts of these results for modules over the Lie algebra $\mathfrak{g}$ of a connected, simply-connected, semi-simple algebraic group G over an algebraically closed field $\Bbbk$ of sufficiently large positive characteristic. In particular we prove that every block of the category of finitely generated modules over the restricted enveloping algebra $(\mathcal{U}\mathfrak{g})_0$ is governed by a Koszul ring, whose dual ring is also related to the representation theory of $\mathfrak{g}$.

The Koszulity of the regular blocks was already proved (under the same assumption on $\Bbbk$) by Andersen, Jantzen and Soergel in [AJS94]. The Koszulity for singular blocks, as well as the information on the dual ring (in all cases) are new, however.

As in [BGS96] we use a geometric picture to prove Koszulity. Over $\mathbb{C}$, the authors of [BGS96] use categories of equivariant perverse sheaves on flag varieties. Over $\Bbbk$ we use as a "substitute" of this tool the localization theory in positive characteristic developed by Bezrukavnikov, Mirković and Rumynin.

0.2

The base of our arguments is a geometric interpretation, due to Mirković, of the classical Koszul duality between symmetric and exterior algebras.

For simplicity, let us first consider the case of a finite dimensional vector space V. Usual Koszul duality (see *e.g.* [BGG78], [BGS96], [GKM93]) relates modules (or dg-modules) over the symmetric algebra $\mathrm{S}(V)$ of V and modules (or dg-modules) over the exterior algebra $\Lambda(V^*)$ of the dual vector space. Geometrically, $\mathrm{S}(V)$ is the ring of functions on the variety V^*. As for $\Lambda(V^*)$, one observes that there exists a quasi-isomorphism of dg-algebras $\Lambda(V^*) \cong \Bbbk \overset{L}{\otimes}_{\mathrm{S}(V^*)} \Bbbk$, where $\Lambda(V^*)$ is equipped with the trivial differential, and the grading such that V^* is in degree -1. Hence $\Lambda(V^*)$ is the ring of functions on the "derived intersection"

$$\{0\} \overset{R}{\cap}_V \{0\},$$

considered as a dg-scheme. An extension of the constructions of [GKM93] yields similarly, if E is a vector bundle over a non-singular variety X and $F \subset E$ is a sub-bundle, a Koszul duality between a certain category of (dg)-sheaves on F and a certain category of (dg)-sheaves on the derived intersection

$$F^\perp \overset{R}{\cap}_{E^*} X,$$

where E^* is the dual vector bundle, $F^\perp \subset E^*$ is the orthogonal of F, and X is regarded as the zero section of E^* (see Theorem 2.3.11 for a precise statement).

0.3

Recall the notation of I.1.1. Here we assume that $p = \mathrm{char}(\Bbbk)$ is bigger than the Coxeter number h of G. Fix a weight $\lambda \in \mathbb{X}$ in the fundamental alcove, and denote similarly the element of $\mathfrak{t}^*$ induced by λ. The results of Bezrukavnikov, Mirković and Rumynin reviewed in chapter I give geometric pictures for the derived categories $\mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,0)}(\mathcal{U}\mathfrak{g})$ and $\mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_0((\mathcal{U}\mathfrak{g})^\lambda)$, as follows (see I.1.2):

$$\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \;\cong\; \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_0((\mathcal{U}\mathfrak{g})^\lambda), \tag{0.3.1}$$

$$\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}) \;\cong\; \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,0)}(\mathcal{U}\mathfrak{g}). \tag{0.3.2}$$

As a first step we derive from (0.3.2) a localization theorem for the category $\mathrm{Mod}^{\mathrm{fg}}_\lambda((\mathcal{U}\mathfrak{g})_0)$ of restricted $\mathcal{U}\mathfrak{g}$-modules with generalized character λ. More precisely, we construct an equivalence

$$\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \;\cong\; \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_\lambda((\mathcal{U}\mathfrak{g})_0),$$

where $\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B}$ is the derived intersection of $\widetilde{\mathfrak{g}}$ and the zero section $\mathcal{B}$ inside the trivial vector bundle $\mathfrak{g}^* \times \mathcal{B}$, and $\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$ is the derived category of coherent dg-sheaves on the Frobenius twist of this derived intersection.

0.4

Under our assumptions on p there is an isomorphism of G-equivariant vector bundles $(\mathfrak{g}^* \times \mathcal{B})^* \cong \mathfrak{g}^* \times \mathcal{B}$. Under this isomorphism, $\widetilde{\mathfrak{g}}$ identifies with the orthogonal of $\widetilde{\mathcal{N}} \subset \mathfrak{g}^* \times \mathcal{B}$. Hence the Koszul duality of 0.2 yields a duality between certain dg-sheaves on $\widetilde{\mathcal{N}}^{(1)}$ and on the derived intersection $(\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}$. Now observe that there is an inclusion $\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^\lambda) \hookrightarrow \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)})$, induced by equivalence (0.3.1). Using the results alluded to in 0.2, we obtain categories $\mathcal{C}^{\mathrm{gr}}$, $\mathcal{D}^{\mathrm{gr}}$, endowed with auto-equivalences denoted $\langle 1 \rangle$ (the *internal shift*), an equivalence $\kappa : \mathcal{C}^{\mathrm{gr}} \xrightarrow{\sim} \mathcal{D}^{\mathrm{gr}}$, and a diagram

$$\begin{tikzcd}
 & \mathcal{C}^{\mathrm{gr}} \arrow[r, "\kappa", "\sim"'] \arrow[d, "\mathrm{For}"'] & \mathcal{D}^{\mathrm{gr}} \arrow[d, "\mathrm{For}"] \\
\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^\lambda) \arrow[r, hook] & \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)}) & \mathcal{D}^b\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0).
\end{tikzcd}$$

Recall that, by a celebrated theorem of Curtis ([Cur60]) and by the decription of the Harish-Chandra center $\mathfrak{Z}_{\mathrm{HC}}$ (see I.1.2), the simple objects in the categories $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^\lambda)$ and $\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$ are the (restrictions of the) simple G-modules $L(\mu)$ for $\mu \in \mathbb{X}$ dominant restricted, in the orbit of λ under the dot-action of the extended affine Weyl group W'_{aff}. The category $\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$ is the category of finitely generated modules over the finite dimensional algebra $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$ (the block of $(\mathcal{U}\mathfrak{g})_0$ associated to λ). We denote by $P(\mu)$ the projective cover of $L(\mu)$ in this category. The objects $L(\mu)$ can be lifted to the category $\mathcal{C}^{\mathrm{gr}}$, uniquely up to the action of the shift $\langle 1 \rangle$. The same is true for the objects $P(\mu)$ and the category $\mathcal{D}^{\mathrm{gr}}$.

Consider the element $\tau_0 := t_\rho \cdot w_0 \in W'_{\mathrm{aff}}$, where t_ρ is the translation by ρ, and w_0 is the longest element of W. Then the key-point of our reasoning is the following (see Theorem 4.4.3 and subsection 8.1):

> Assume $p \gg 0$. Then there exists a unique choice of the lifts $L^{\mathrm{gr}}(\mu) \in \mathcal{C}^{\mathrm{gr}}$, $P^{\mathrm{gr}}(\mu) \in \mathcal{D}^{\mathrm{gr}}$ such that if $w \in W'_{\mathrm{aff}}$ and $w \bullet \lambda$ is dominant restricted, then $\kappa(L^{\mathrm{gr}}(w \bullet \lambda)) \cong P^{\mathrm{gr}}(\tau_0 w \bullet \lambda)$.

In other words, our "geometric" Koszul duality exchanges semi-simple and projective modules.

This result was supported by the calculations in small ranks of sections I.2 and I.3.

0.5

Our proof of this key-point relies on the study of "geometric counterparts" of the reflection functors $\mathfrak{R}_\delta^{\mathrm{gr}} : \mathcal{D}^{\mathrm{gr}} \to \mathcal{D}^{\mathrm{gr}}$ (here δ is an affine simple root), which send (lifts of) projective modules to (lifts of) projective modules. We identify the "Koszul dual" (*i.e.* the conjugate by κ) of these functors, which are related to some functors $\mathfrak{S}_\delta^{\mathrm{gr}}$ which send (lifts of) some semi-simple modules to (lifts of) semi-simple modules (see Theorem 8.2.1). Then we only have to check our key-point when $\ell(w) = 0$, which can be done directly (and explicitely).

To prove the "semi-simplicity" of the functors $\mathfrak{S}_{\delta}^{\mathrm{gr}}$ we use Lusztig's conjecture on the characters of simple G-modules, see [Lus80b] (or rather an equivalent formulation of this conjecture due to Andersen, see [And86]). Recall that, by the previously cited work of Andersen-Jantzen-Soergel ([AJS94]), combined with works of Kazhdan-Lusztig ([KL93a], [KL93b], [KL94a], [KL94b], [Lus94]) and Kashiwara-Tanisaki ([KT95], [KT96]), (see also [ABG04] or [Fie07] for other approaches), this conjecture is true for p sufficiently large (with no explicit bound). This explains our restriction on p.

Let us remark that related ideas (in particular, a contruction of graded versions of translation functors) were considered by Stroppel in [Str03] for the category $\mathcal{O}$ in characteristic 0. However, our techniques are completely different.

0.6

We derive from the key-point of 0.4 the Koszulity of regular blocks of $(\mathcal{U}\mathfrak{g})_0$. For this we use a general criterion for a graded ring to be Morita equivalent to a Koszul ring, proved in Theorem 9.2.1. More precisely we obtain the following result (see Theorem 9.5.1):

There exists a (unique) grading on the block $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$ which makes it a Koszul ring. The Koszul dual ring controls the category $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^{\lambda})$.

Hence, from a "geometric" Koszul duality between the dg-schemes $\widetilde{\mathcal{N}}^{(1)}$ and $(\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}$ we derive an "algebraic" Koszul duality between the abelian categories $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^{\lambda})$ and $\mathrm{Mod}_{\lambda}^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$.

0.7

Finally we consider a "parabolic analogue" of our geometric duality, where $\mathcal{B}$ is replaced by a partial flag variety $\mathcal{P}$. We prove a version of our restricted localization theorem for singular weights (see Theorem 3.3.15). Then we derive from our key-point (see 0.4) a version of it for this "parabolic" duality, and we deduce Koszulity of singular blocks of $(\mathcal{U}\mathfrak{g})_0$ (see Theorem 10.3.1). In this case the Koszul dual ring is related to a quotient of $\mathcal{U}\mathfrak{g}$ introduced in [BMR08, §1.10].

In particular, it follows that, for $p \gg 0$, the ring $(\mathcal{U}\mathfrak{g})_0$ can be endowed with a (unique) Koszul grading, *i.e.* a grading which makes it a Koszul ring (see Corollary 10.3.2). This fact was conjectured (for all $p > h$) by Soergel in [Soe94].

0.8

Another interest of our key-point of 0.4 is that it gives information on the complexes of coherent sheaves corresponding to simple and projective $\mathcal{U}\mathfrak{g}$-modules under equivalences (0.3.1) and (0.3.2). (The question of computing these objects was raised in [BMR06, 1.5.1].) Namely, the objects corresponding to indecomposable projectives and to simples are related by the simple geometric construction of 0.2. Our proof also provides a way to "generate" these objects. Namely, to compute them it suffices to apply explicit functors to explicit sheaves, and to take direct factors. In practice these computations are very difficult, however.

0.9 Organization of the chapter

In section 1 we develop the necessary background on derived categories of sheaves of dg-modules over sheaves of dg-algebras, extending results of [BL94] and [Spa88]. We also introduce some notions related to dg-schemes (in the sense of [CFK01]).

In section 2 we give a geometric version of Koszul duality (due to Mirković), and study how this duality behaves under proper flat base change, and with respect to sub-bundles.

In section 3 we prove a localization theorem for restricted $\mathcal{U}\mathfrak{g}$-modules, as an extension of the results of [BMR08], [BMR06].

In section 4 we state a version of our key-point. Sections 5 to 8 are devoted to the proof of this theorem.

In section 5 we introduce some useful tools for our study, in particular some braid group actions, using the main result of chapter II.

In section 6 we study the projective $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$-modules and their geometric counterparts, and their behaviour under the reflection functors. Here and below, $\lambda \in \mathbb{X}$ is a regular integral character.

Similarly, in section 7 we study the simple restricted $(\mathcal{U}\mathfrak{g})^{\lambda}$-modules and their geometric counterparts, and their behaviour under the "semi-simple" functors $\mathfrak{S}_{\delta}$.

In section 8 we finally prove that the "geometric" Koszul duality exchanges the indecomposable projective $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$-modules and the simple restricted $(\mathcal{U}\mathfrak{g})^{\lambda}$-modules.

In section 9 we derive the fact that there is an "algebraic" Koszul duality relating $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$-modules and $(\mathcal{U}\mathfrak{g})^{\lambda}$-modules with generalized trivial Frobenius character.

Finally, in section 10 we extend some of our results to singular characters. In particular we prove Koszulity of singular blocks of $(\mathcal{U}\mathfrak{g})_0$.

1 Sheaves of dg-algebras and dg-modules

In this section we extend results on dg-algebras and ringed spaces (see [BL94] and [Spa88]) to the case of a sheaf of dg-algebras on a ringed space. Most of these extensions are straightforward, but certain results require some special care, especially concerning the existence of resolutions. We fix a commutative ringed space $(X, \mathcal{O}_X)$, and write simply $\otimes$ for $\otimes_{\mathcal{O}_X}$.

1.1 Definitions

Let $\mathcal{A} = \bigoplus_{p \in \mathbb{Z}} \mathcal{A}^p$ be a sheaf of $\mathbb{Z}$-graded $\mathcal{O}_X$-algebras on X. Denote by $\mu_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ the multiplication map.

Definition 1.1.1. $\mathcal{A}$ is a *sheaf of dg-algebras* if it is provided with an endomorphism of $\mathcal{O}_X$-modules $d_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$, of degree 1, such that $d_{\mathcal{A}} \circ d_{\mathcal{A}} = 0$, and satisfying the following formula on $\mathcal{A}^p \otimes \mathcal{A}$, for any $p \in \mathbb{Z}$:

$$d_{\mathcal{A}} \circ \mu_{\mathcal{A}} = \mu_{\mathcal{A}} \circ (d_{\mathcal{A}} \otimes \mathrm{Id}_{\mathcal{A}}) + (-1)^p \mu_{\mathcal{A}} \circ (\mathrm{Id}_{\mathcal{A}^p} \otimes d_{\mathcal{A}}).$$

A *morphism of sheaves of dg-algebras* $\phi : \mathcal{A} \to \mathcal{B}$ is a morphism of sheaves of graded algebras commuting with the differentials.

A *sheaf of dg-modules* over $\mathcal{A}$ (in short: $\mathcal{A}$-dg-module) is a sheaf of graded left $\mathcal{A}$-modules $\mathcal{F}$ on X, provided with an endomorphism of $\mathcal{O}_X$-modules $d_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$, of degree 1, such that $d_{\mathcal{F}} \circ d_{\mathcal{F}} = 0$, and satisfying the following formula on $\mathcal{A}^p \otimes \mathcal{F}$, for any $p \in \mathbb{Z}$, where $\alpha_{\mathcal{F}} : \mathcal{A} \otimes \mathcal{F} \to \mathcal{F}$ is the action map:

$$d_{\mathcal{F}} \circ \alpha_{\mathcal{F}} = \alpha_{\mathcal{F}} \circ (d_{\mathcal{A}} \otimes \mathrm{Id}_{\mathcal{F}}) + (-1)^p \alpha_{\mathcal{F}} \circ (\mathrm{Id}_{\mathcal{A}^p} \otimes d_{\mathcal{F}}).$$

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of dg-modules over $\mathcal{A}$, a *morphism of sheaves of dg-modules* $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of graded $\mathcal{A}$-modules commuting with the differentials.

We will always consider $\mathcal{O}_X$ as a sheaf of dg-algebras concentrated in degree 0, provided with the zero differential. In the rest of this section we fix a sheaf of dg-algebras $\mathcal{A}$.

We denote by $\mathcal{C}(X, \mathcal{A})$ (or sometimes simply $\mathcal{C}(\mathcal{A})$) the category of sheaves of dg-modules over $\mathcal{A}$. The *translation functor* $[1] : \mathcal{C}(X, \mathcal{A}) \to \mathcal{C}(X, \mathcal{A})$ is defined as usual to be the auto-equivalence of $\mathcal{C}(X, \mathcal{A})$ given by:

$$(\mathcal{F}[1])^p = \mathcal{F}^{p+1}, \quad d_{\mathcal{F}[1]} = -d_{\mathcal{F}},$$

and the $\mathcal{A}$-module structure is twisted as follows: on $\mathcal{A}^p \otimes \mathcal{F}[1]$,

$$\alpha_{\mathcal{F}[1]} = (-1)^p \alpha_{\mathcal{F}}.$$

Again as usual, two morphisms $\phi, \psi : \mathcal{F} \to \mathcal{G}$ in $\mathcal{C}(X, \mathcal{A})$ are said to be *homotopic* if there exists a morphism of graded $\mathcal{A}$-modules $h : \mathcal{F} \to \mathcal{G}[-1]$ (not necessarily commuting with the differentials) such that

$$\phi - \psi = h \circ d_{\mathcal{F}} + d_{\mathcal{G}} \circ h.$$

We define then the homotopy category $\mathcal{H}(X, \mathcal{A})$ whose objects are those of $\mathcal{C}(X, \mathcal{A})$, and whose morphisms are obtained by quotienting the morphisms in $\mathcal{C}(X, \mathcal{A})$ by the homotopy relation.

If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism in $\mathcal{C}(X, \mathcal{A})$ or $\mathcal{H}(X, \mathcal{A})$, we define its *cone* to be the graded $\mathcal{A}$-module $\mathrm{Cone}(\phi) := \mathcal{G} \oplus \mathcal{F}[1]$, provided with the differential given in degree n by the matrix

$$\begin{pmatrix} d_{\mathcal{G}}^n & \phi^{n+1} \\ 0 & d_{\mathcal{F}[1]}^n \end{pmatrix}.$$

We define an exact triangle in $\mathcal{H}(X, \mathcal{A})$ to be a triangle isomorphic to a triangle of the form

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \longrightarrow \mathrm{Cone}(\phi) \longrightarrow \mathcal{F}[1].$$

Provided with these exact triangles and the translation functor [1] defined above, $\mathcal{H}(X, \mathcal{A})$ has a structure of a triangulated category.

If $\mathcal{F}$ is an object of $\mathcal{C}(X, \mathcal{A})$ or $\mathcal{H}(X, \mathcal{A})$, we define its cohomology to be the graded sheaf of $\mathcal{O}_X$-modules $H(\mathcal{F}) = \mathrm{Ker}(d_{\mathcal{F}})/\mathrm{Im}(d_{\mathcal{F}})$. A dg-module $\mathcal{F}$ is said to be *acyclic* if $H(\mathcal{F}) = 0$. A morphism $\phi : \mathcal{F} \to \mathcal{G}$ in $\mathcal{C}(X, \mathcal{A})$ or $\mathcal{H}(X, \mathcal{A})$ is said to be a *quasi-isomorphism* if it induces an isomorphism $H(\phi) : H(\mathcal{F}) \xrightarrow{\sim} H(\mathcal{G})$. This is equivalent to the property that $\mathrm{Cone}(\phi)$ is acyclic. Finally we define the derived category $\mathcal{D}(X, \mathcal{A})$ to be the localization of $\mathcal{H}(X, \mathcal{A})$ with respect to quasi-isomorphisms. It inherits a structure of a triangulated category from $\mathcal{H}(X, \mathcal{A})$.

We define similarly the category $\mathcal{C}^r(X, \mathcal{A})$ of sheaves of right $\mathcal{A}$-dg-modules, its homotopy category $\mathcal{H}^r(X, \mathcal{A})$ and its derived category $\mathcal{D}^r(X, \mathcal{A})$. We define the opposite sheaf of dg-algebras $\mathcal{A}^{\mathrm{op}}$ which equals $\mathcal{A}$ as a sheaf of $\mathcal{O}_X$-dg-modules, and where the multiplication is given on $(\mathcal{A}^{\mathrm{op}})^p \otimes (\mathcal{A}^{\mathrm{op}})^q$ by the composition

$$\begin{array}{ccccc} \mathcal{A}^p \otimes \mathcal{A}^q & \xrightarrow{\sim} & \mathcal{A}^q \otimes \mathcal{A}^p & \xrightarrow{\mu_{\mathcal{A}}} & \mathcal{A}^{p+q} \\ a \otimes b & \mapsto & (-1)^{pq} b \otimes a & & \end{array}.$$

As usual there is a natural equivalence of categories

$$\mathcal{C}^r(X, \mathcal{A}) \xrightarrow{\sim} \mathcal{C}(X, \mathcal{A}^{\mathrm{op}}) \tag{1.1.2}$$

sending the object $\mathcal{F} \in \mathcal{C}^r(\mathcal{A})$ to the object of $\mathcal{C}(\mathcal{A}^{\mathrm{op}})$ which equals $\mathcal{F}$ as an $\mathcal{O}_X$-dg-module, and where the action of $(\mathcal{A}^{\mathrm{op}})^p$ on $\mathcal{F}^q$ is given by

$$\begin{array}{cccccc} (\mathcal{A}^{\mathrm{op}})^p \otimes \mathcal{F} & = & \mathcal{A}^p \otimes \mathcal{F}^q & \xrightarrow{\sim} & \mathcal{F}^q \otimes \mathcal{A}^p & \xrightarrow{\alpha_{\mathcal{F}}} & \mathcal{F}^{p+q} \\ & & a \otimes f & \mapsto & (-1)^{pq} f \otimes a & & \end{array}.$$

A sheaf of dg-algebras $\mathcal{A}$ is said to be *graded-commutative* if the identity map $\mathrm{Id} : \mathcal{A} \to \mathcal{A}^{\mathrm{op}}$ is an isomorphism of sheaves of dg-algebras. In this case (1.1.2) gives an equivalence of categories $\mathcal{C}(X, \mathcal{A}) \cong \mathcal{C}^r(X, \mathcal{A})$.

1.2 Hom, Tens and (co)induction

Let $\mathcal{F}$ and $\mathcal{G}$ be objects of $\mathcal{C}(X, \mathcal{A})$. We define the sheaf of $\mathcal{O}_X$*-dg-modules*

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$$

having, as degree p component, the $\mathcal{O}_X$-module of local homomorphisms of graded $\mathcal{A}$-modules $\mathcal{F} \to \mathcal{G}[p]$ (not necessarily commuting with the differentials), and provided with the differential given by

$$d(\phi) = d_{\mathcal{G}} \circ \phi - (-1)^p \phi \circ d_{\mathcal{F}} \tag{1.2.1}$$

if $\phi \in (\mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{G}))^p$. This construction defines a bifunctor

$$\mathcal{H}om_{\mathcal{A}}(-,-) : \mathcal{C}(X,\mathcal{A})^{\mathrm{op}} \times \mathcal{C}(X,\mathcal{A}) \to \mathcal{C}(X,\mathcal{O}_X).$$

One easily checks that $\mathcal{H}om_{\mathcal{A}}(-,-)$ preserves homotopies, and thus defines a bifunctor

$$\mathcal{H}om_{\mathcal{A}}(-,-) : \mathcal{H}(X,\mathcal{A})^{\mathrm{op}} \times \mathcal{H}(X,\mathcal{A}) \to \mathcal{H}(X,\mathcal{O}_X).$$

In case $\mathcal{A}$ is graded-commutative, this even defines naturally a bifunctor

$$\mathcal{H}om_{\mathcal{A}}(-,-) : \mathcal{H}(X,\mathcal{A})^{\mathrm{op}} \times \mathcal{H}(X,\mathcal{A}) \to \mathcal{H}(X,\mathcal{A}).$$

We also define the functor $\mathrm{Hom}_{\mathcal{A}}(-,-)$, from $\mathcal{C}(X,\mathcal{A})^{\mathrm{op}} \times \mathcal{C}(X,\mathcal{A})$ to the category of complexes of abelian groups, by putting

$$(\mathrm{Hom}_{\mathcal{A}}(\mathcal{F},\mathcal{G}))^i := \Gamma(X, (\mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{G}))^i),$$

the differential being that of (1.2.1). As usual, the group $\mathrm{Hom}_{\mathcal{C}(X,\mathcal{A})}(\mathcal{F},\mathcal{G})$ is the kernel of the differential d^0 on $(\mathrm{Hom}_{\mathcal{A}}(\mathcal{F},\mathcal{G}))^0$, and $\mathrm{Hom}_{\mathcal{H}(X,\mathcal{A})}(\mathcal{F},\mathcal{G}) \cong H^0(\mathrm{Hom}_{\mathcal{A}}(\mathcal{F},\mathcal{G}))$.

Let $\mathcal{F}$ be an object of $\mathcal{C}^r(X,\mathcal{A})$, and let $\mathcal{G}$ be an object of $\mathcal{C}(X,\mathcal{A})$. We define the sheaf of $\mathcal{O}_X$-dg-modules $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$, graded in the natural way, and having the differential given on local sections of $\mathcal{F}^p \otimes_{\mathcal{A}} \mathcal{G}$ by

$$d(f \otimes g) = d(f) \otimes g + (-1)^p f \otimes d(g).$$

This construction defines a bifunctor

$$(- \otimes_{\mathcal{A}} -) : \mathcal{C}^r(X,\mathcal{A}) \times \mathcal{C}(X,\mathcal{A}) \to \mathcal{C}(X,\mathcal{O}_X).$$

One easily checks that $(- \otimes_{\mathcal{A}} -)$ preserves homotopies, and thus defines a bifunctor

$$(- \otimes_{\mathcal{A}} -) : \mathcal{H}^r(X,\mathcal{A}) \times \mathcal{H}(X,\mathcal{A}) \to \mathcal{H}(X,\mathcal{O}_X).$$

As usual the tensor product is associative.

Let us define the induction functor in the usual way:

$$\mathrm{Ind} : \left\{ \begin{array}{ccc} \mathcal{C}(X,\mathcal{O}_X) & \to & \mathcal{C}(X,\mathcal{A}) \\ \mathcal{F} & \mapsto & \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F}. \end{array} \right.$$

This functor is a left adjoint to the forgetful functor $\mathrm{For} : \mathcal{C}(X,\mathcal{A}) \to \mathcal{C}(X,\mathcal{O}_X)$. More precisely, for $\mathcal{F}$ in $\mathcal{C}(X,\mathcal{O}_X)$ and $\mathcal{G}$ in $\mathcal{C}(X,\mathcal{A})$, we have a functorial isomorphism of $\mathcal{O}_X$-dg-modules:

$$\mathcal{H}om_{\mathcal{A}}(\mathrm{Ind}(\mathcal{F}),\mathcal{G}) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathrm{For}(\mathcal{G})), \tag{1.2.2}$$

and thus, taking the global sections and then the kernels of d^0, one obtains:

$$\mathrm{Hom}_{\mathcal{C}(X,\mathcal{A})}(\mathrm{Ind}(\mathcal{F}),\mathcal{G}) \cong \mathrm{Hom}_{\mathcal{C}(X,\mathcal{O}_X)}(\mathcal{F},\mathcal{G}).$$

The functor Ind also induces a functor $\mathcal{H}(X, \mathcal{O}_X) \to \mathcal{H}(X, \mathcal{A})$, which is left adjoint to the forgetful functor $\mathcal{H}(X, \mathcal{A}) \to \mathcal{H}(X, \mathcal{O}_X)$. For later use, let us remark that the adjunction morphism $\mathrm{Ind}(\mathcal{F}) \to \mathcal{F}$ is surjective for $\mathcal{F} \in \mathcal{C}(X, \mathcal{A})$.

Now we define the coinduction functor

$$\mathrm{Coind}: \left\{ \begin{array}{ccc} \mathcal{C}(X, \mathcal{O}_X) & \to & \mathcal{C}(X, \mathcal{A}) \\ \mathcal{G} & \mapsto & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}, \mathcal{G}) \end{array} \right.$$

(and similarly for the homotopy categories) where the grading and differential are defined as in (1.2.1), and the action of $\mathcal{A}$ is given on local sections by

$$(\alpha \cdot \phi)(\gamma) = (-1)^{\deg(\alpha)\deg(\phi) + \deg(\alpha)\deg(\gamma)} \phi(\gamma\alpha).$$

Let us show that the functor Coind is a right adjoint to the forgetful functor $\mathcal{C}(X, \mathcal{A}) \to \mathcal{C}(X, \mathcal{O}_X)$. Let $\mathcal{F}$ be an $\mathcal{A}$-dg-module, and $\mathcal{G}$ be an $\mathcal{O}_X$-dg-module. We define the morphism

$$\phi : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathrm{Coind}(\mathcal{G}))$$

by the following formula, where λ, resp. f, resp. α is a local section of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, resp. $\mathcal{F}$, resp. $\mathcal{A}$: $\phi(\lambda)(f)(\alpha) = (-1)^{\deg(\alpha)\deg(f)} \lambda(\alpha f)$. We also define the morphism:

$$\psi : \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathrm{Coind}(\mathcal{G})) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

by the formula $\psi(\mu)(f) = \mu(f)(1_{\mathcal{A}})$. The proof of the next lemma is a straightforward computation, left to the reader.

Lemma 1.2.3. *ϕ and ψ are inverse isomorphisms of $\mathcal{O}_X$-dg-modules. In particular, they induce isomorphisms of complexes of abelian groups, respectively of abelian groups:*

$$\begin{gathered} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathrm{Coind}(\mathcal{G})), \\ \mathrm{Hom}_{\mathcal{C}(X, \mathcal{O}_X)}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{C}(X, \mathcal{A})}(\mathcal{F}, \mathrm{Coind}(\mathcal{G})), \\ \mathrm{Hom}_{\mathcal{H}(X, \mathcal{O}_X)}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{H}(X, \mathcal{A})}(\mathcal{F}, \mathrm{Coind}(\mathcal{G})). \end{gathered}$$

For later use, let us remark that the adjunction morphism $\mathcal{G} \to \mathrm{Coind}(\mathcal{G})$ is injective, for $\mathcal{G} \in \mathcal{C}(X, \mathcal{A})$.

1.3 Existence of K-flat and K-injective resolutions

To ensure the existence of the derived functors of the usual functors, we have to show that there are enough objects in the category $\mathcal{C}(\mathcal{A})$ having nice properties. For this we follow Spaltenstein's approach ([Spa88]).

Definition 1.3.1. Let $\mathcal{F}$ be an object of $\mathcal{C}(\mathcal{A})$. We say that $\mathcal{F}$ is *K-injective* if one of the following equivalent properties holds:

(i) For every object $\mathcal{G}$ of $\mathcal{C}(\mathcal{A})$, $\mathrm{Hom}_{\mathcal{H}(\mathcal{A})}(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{G}, \mathcal{F})$;

(ii) For every object $\mathcal{G}$ of $\mathcal{C}(\mathcal{A})$ such that $H(\mathcal{G}) = 0$, $H(\mathrm{Hom}_{\mathcal{A}}(\mathcal{G}, \mathcal{F})) = 0$.

In (ii), $H(\mathrm{Hom}_{\mathcal{A}}(\mathcal{G},\mathcal{F}))$ is the cohomology as a complex of abelian groups. For a proof of the equivalence of these conditions, see [BL94, 10.12.2.2].

We will also consider the analogue of a flat resolution.

Definition 1.3.2. An object $\mathcal{F}$ of $\mathcal{C}(\mathcal{A})$ is said to be *K-flat* if for every object $\mathcal{G}$ of $\mathcal{C}^r(\mathcal{A})$ such that $H(\mathcal{G})=0$, we have $H(\mathcal{G}\otimes_{\mathcal{A}}\mathcal{F})=0$.

Easy applications of the basic properties of induction and coinduction functors give the following two lemmas:

Lemma 1.3.3. *If $\mathcal{F}$ is a K-flat $\mathcal{O}_X$-dg-module, then* $\mathrm{Ind}(\mathcal{F})$ *is a K-flat $\mathcal{A}$-dg-module. If $\mathcal{G}$ is a K-injective $\mathcal{O}_X$-dg-module, then* $\mathrm{Coind}(\mathcal{G})$ *is a K-injective $\mathcal{A}$-dg-module.*

Lemma 1.3.4. *Assume $\mathcal{A}$ is K-flat as an $\mathcal{O}_X$-dg-module. Then every K-injective $\mathcal{A}$-dg-module is also K-injective as an $\mathcal{O}_X$-dg-module. Similarly, every K-flat $\mathcal{A}$-dg-module is also K-flat as an $\mathcal{O}_X$-dg-module.*

Let us prove that there exist enough K-flat modules in $\mathcal{C}(X,\mathcal{A})$. The case $\mathcal{A}=\mathcal{O}_X$ is treated in [Spa88], and will be the base of our proofs.

If $\mathcal{M}$ is a complex of sheaves, we denote by $Z(\mathcal{M})$ the graded sheaf $\mathrm{Ker}(d_{\mathcal{M}})$.

Theorem 1.3.5. *For every sheaf of $\mathcal{A}$-dg-modules $\mathcal{F}$, there exists a K-flat sheaf of $\mathcal{A}$-dg-modules $\mathcal{P}$ and a quasi-isomorphism $\mathcal{P}\xrightarrow{qis}\mathcal{F}$.*

Proof. First, let us consider $\mathcal{F}$ as an $\mathcal{O}_X$-dg-module. By [Spa88, 5.6], there exists a K-flat $\mathcal{O}_X$-dg-module $\mathcal{Q}_0$ and a surjective quasi-isomorphism of $\mathcal{O}_X$-dg-modules $\mathcal{Q}_0\twoheadrightarrow\mathcal{F}$. Thus there exists a surjective morphism of $\mathcal{A}$-dg-modules

$$\mathcal{P}_0:=\mathrm{Ind}(\mathcal{Q}_0)\twoheadrightarrow\mathrm{Ind}(\mathcal{F})\twoheadrightarrow\mathcal{F},$$

and the $\mathcal{A}$-dg-module $\mathcal{P}_0$ is K-flat, by Lemma 1.3.3. The induced morphism $Z(\mathcal{P}_0)\to Z(\mathcal{F})$ is also surjective. This follows from the fact that the morphism $Z(\mathcal{Q}_0)\to Z(\mathcal{F})$ is surjective, because $\mathcal{Q}_0\to\mathcal{F}$ is a surjective quasi-isomorphism.

Doing the same construction for the kernel of the morphism $\mathcal{P}_0\to\mathcal{F}$, and repeating, we obtain an exact sequence of $\mathcal{A}$-dg-modules

$$\cdots\to\mathcal{P}_1\to\mathcal{P}_0\to\mathcal{F}\to 0$$

where each $\mathcal{P}_p$ is K-flat, and such that the induced sequence

$$\cdots\to Z(\mathcal{P}_1)\to Z(\mathcal{P}_0)\to Z(\mathcal{F})\to 0$$

is also exact. Now we take the $\mathcal{A}$-dg-module $\mathcal{P}:=\mathrm{Tot}^{\oplus}(\cdots\to\mathcal{P}_1\to\mathcal{P}_0\to 0\to\cdots)$, where $\mathcal{P}_p$ is in horizontal degree $-p$. It is K-flat, as the direct limit of the K-flat $\mathcal{A}$-dg-modules $\mathcal{P}_{\leq p}:=\mathrm{Tot}^{\oplus}(\cdots\to 0\to\mathcal{P}_p\to\cdots\to\mathcal{P}_0\to 0\to\cdots)$ (see [Spa88, 5.4.(c)]). Now we prove that the natural morphism $\mathcal{P}\to\mathcal{F}$ is a quasi-isomorphism, *i.e.* that the complex $\mathcal{X}:=\mathrm{Tot}^{\oplus}(\cdots\to\mathcal{P}_1\to\mathcal{P}_0\to\mathcal{F}\to 0\to\cdots)$, where $\mathcal{F}$ is in horizontal degree 1, is acyclic.

The argument for this is borrowed from [Kel94, 3.3], [Kel00]. We put $\mathcal{P}_{-1} := \mathcal{F}$, and $\mathcal{P}_p = 0$ if $p < -1$. Consider, for $m \geq 1$, the double complex of $\mathcal{O}_X$-modules $\mathcal{X}_m$ defined by $(\mathcal{X}_m)^{i,j} = 0$ if $j \notin [-m, m]$, $(\mathcal{X}_m)^{i,j} = (\mathcal{P}_{-i})^j$ if $j \in [-m, m-1]$, and $(\mathcal{X}_m)^{i,m} = Z(\mathcal{P}_{-i})^m$. Then $\mathcal{X}$ is the direct limit of the complexes $\mathrm{Tot}^{\oplus}(\mathcal{X}_m)$, which are acyclic because they admit a finite filtration with acyclic subquotients. Hence $\mathcal{X}$ is acyclic. □

We will also need the following result, which is borrowed from [Spa88, 5.7]:

Lemma 1.3.6. *If $\mathcal{P}$ in $\mathcal{C}(\mathcal{A})$ is K-flat and acyclic, then for any $\mathcal{F}$ in $\mathcal{C}^r(\mathcal{A})$ the $\mathcal{O}_X$-dg-module $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{P}$ is acyclic.*

Proof. Let $\mathcal{Q}$ be a K-flat left resolution of $\mathcal{F}$ (in $\mathcal{C}^r(\mathcal{A}) \cong \mathcal{C}(\mathcal{A}^{\mathrm{op}})$). Since $\mathcal{P}$ is K-flat, $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{P}$ is quasi-isomorphic to $\mathcal{Q} \otimes_{\mathcal{A}} \mathcal{P}$, which is acyclic since $\mathcal{Q}$ is K-flat and $\mathcal{P}$ acyclic. □

From now on in this section we make the following assumptions:

(†) All our topological spaces are noetherian of finite dimension.
(††) All our dg-algebras are concentrated in non-positive degrees.

These assumptions are needed for our proofs and sufficient for our applications, but we hope they are not essential. In order to construct resolutions by K-injective $\mathcal{A}$-dg-modules, we begin with the case of bounded below dg-modules.

Lemma 1.3.7. *For every bounded-below $\mathcal{A}$-dg-module $\mathcal{F}$, there exists a quasi-isomorphism of $\mathcal{A}$-dg-modules $\mathcal{F} \xrightarrow{\mathrm{qis}} \mathcal{I}$, where $\mathcal{I}$ is a K-injective $\mathcal{A}$-dg-module, bounded below with the same bound as $\mathcal{F}$ and such that $\mathcal{I}^p$ is a flabby sheaf for every $p \in \mathbb{Z}$.*

Proof. Let us first consider $\mathcal{F}$ as a sheaf of $\mathcal{O}_X$-dg-modules. As it is bounded below, there exists a bounded below $\mathcal{O}_X$-dg-module $\mathcal{J}_0$ (with the same bound as $\mathcal{F}$), all of whose components are injective $\mathcal{O}_X$-modules, and an injective morphism $\phi : \mathcal{F} \hookrightarrow \mathcal{J}_0$. Then $\mathcal{J}_0$ is a K-injective $\mathcal{O}_X$-dg-module by [Spa88, 1.2, 2.2.(c), 2.5]. By Lemma 1.3.3, $\mathcal{I}_0 := \mathrm{Coind}(\mathcal{J}_0)$ is a K-injective $\mathcal{A}$-dg-module, and one obtains an injective morphism of $\mathcal{A}$-dg-modules

$$\mathcal{F} \hookrightarrow \mathrm{Coind}(\mathcal{F}) \hookrightarrow \mathcal{I}_0.$$

This module is bounded below, again with the same bound (because $\mathcal{A}$ is non-positively graded), and its graded components are flabby (use the classical fact that if $\mathcal{E}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, with $\mathcal{G}$ injective, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G})$ is flabby, see [KS90, II.2.4.6.(vii)], and the fact that a product of flabby sheaves is flabby). Let $\mathcal{X}_0$ denote the cokernel of this morphism. We have an exact sequence of $\mathcal{A}$-dg-modules $0 \to \mathcal{F} \xrightarrow{i} \mathcal{I}_0 \xrightarrow{p} \mathcal{X}_0 \to 0$. Repeating the same construction for $\mathcal{X}_0$, and then again and again, we obtain an exact sequence of $\mathcal{A}$-dg-modules (bounded below with the same bound for all the modules)

$$0 \to \mathcal{F} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \mathcal{I}_2 \to \cdots$$

where each $\mathcal{I}_p$ is K-injective and has flabby components.

Let us consider the double complex defined by

$$\mathcal{N}^{pq} := \begin{cases} \mathcal{I}_p^q & \text{if } p \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and define the $\mathcal{A}$-dg-module $\mathcal{I} := \mathrm{Tot}^{\oplus}(\mathcal{N})$. This module is the inverse limit of the $\mathcal{A}$-dg-modules $\mathcal{K}_p := \mathrm{Tot}^{\oplus}(\cdots \to 0 \to \mathcal{I}_0 \to \cdots \to \mathcal{I}_p \to 0 \to \cdots)$ for $p \geq 0$ (all the direct sums involved are finite, hence commute with inverse limits). For each $p \geq 0$, $\mathcal{K}_p$ is a K-injective $\mathcal{A}$-dg-module (because it has a finite filtration with K-injective subquotients). Moreover, the morphisms $\mathcal{K}_{p+1} \to \mathcal{K}_p$ are surjective, and split as morphisms of graded $\mathcal{A}$-modules. Hence this inverse system is "special" in the sense of [Spa88, 2.1]. We deduce that $\mathcal{I}$ is a K-injective $\mathcal{A}$-dg-module (use [Spa88, 2.3, 2.4]). This module also has flabby components (because a finite sum of flabby sheaves is flabby). Now we only have to show that the natural morphism $\mathcal{F} \to \mathcal{I}$ is a quasi-isomorphism, *i.e.* that the $\mathcal{O}_X$-dg-module $\mathrm{Tot}^{\oplus}(\cdots \to 0 \to \mathcal{F} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \mathcal{I}_2 \to \cdots)$ is acyclic.

It suffices to show that for any $x \in X$ the complex $\mathrm{Tot}^{\oplus}(\ldots \to 0 \to \mathcal{F}_x \to (\mathcal{I}_0)_x \to (\mathcal{I}_1)_x \to (\mathcal{I}_2)_x \to \ldots)$ is acyclic. This follows easily from the usual spectral sequence of a first quadrant double complex. □

Now we can treat the general case. Recall the classical definition of the truncation functors: if M is a complex of objects of an abelian category, for every $n \in \mathbb{Z}$ we define the complex

$$\tau_{\geq n} M := (\cdots \to 0 \to M^n/(\mathrm{Im}\ d^{n-1}) \to M^{n+1} \to \cdots).$$

The natural morphism $M \to \tau_{\geq n} M$ induces an isomorphism on cohomology groups H^m for $m \geq n$, and $H^m(\tau_{\geq n} M) = 0$ for $m < n$. For any n we have a surjection $\tau_{\geq n} M \to \tau_{\geq n+1} M$, whose kernel is quasi-isomorphic to $H^n(M)[-n]$. Because of our assumption (††), this definition is still meaningful (and has the same properties) for $\mathcal{A}$-dg-modules.

Theorem 1.3.8. *For every $\mathcal{A}$-dg-module $\mathcal{F}$, there exists a quasi-isomorphism of $\mathcal{A}$-dg-modules $\mathcal{F} \xrightarrow{qis} \mathcal{I}$ where $\mathcal{I}$ is a K-injective $\mathcal{A}$-dg-module.*

Proof. Using the preceding lemma, the construction of [Spa88, 3.7] generalizes: there exists an inverse system of morphisms of $\mathcal{A}$-dg-modules

$$f_n : \tau_{\geq -n} \mathcal{F} \xrightarrow{\mathrm{qis}} \mathcal{I}_n$$

where f_n is a quasi-isomorphism, $\mathcal{I}_n$ is a K-injective $\mathcal{A}$-dg-module with $\mathcal{I}_n^p = 0$ for $p < -n$ and $\mathcal{I}_n^p$ flabby for $p \geq -n$, and, furthermore, the morphisms $\mathcal{I}_{n+1} \to \mathcal{I}_n$ are surjective and split as morphisms of graded $\mathcal{A}$-modules. Then, as in the proof of the previous lemma, $\varprojlim \mathcal{I}_n$ is K-injective. As $\mathcal{F} \cong \varprojlim \tau_{\geq -n} \mathcal{F}$, it remains only to prove that $f := \varprojlim f_n$ is a quasi-isomorphism. For this we can follow the arguments of [Spa88, 3.13]. Indeed, using Grothendieck's vanishing theorem ([Har77, III.2.7]), condition 3.12.(1) of [Spa88] is satisfied with $\mathfrak{B} = \mathrm{Mod}(\mathcal{O}_X)$, and $d_x = \dim(X)$ for any $x \in X$. Moreover, in the proof of [Spa88, 3.13], the fact that the $\mathcal{I}_n$ are K-injective over $\mathcal{O}_X$ is not really needed. In fact, we only need to know that, for every n, the kernel $\mathcal{K}_n$ of the morphism $\mathcal{I}_n \to \mathcal{I}_{n-1}$ is a resolution of $\mathcal{H}^{-n}(\mathcal{F})[n]$ which is acyclic for the functors $\Gamma(U,-)$ for every open $U \subset X$. In our case, $\mathcal{K}_n$ is a flabby resolution of $\mathcal{H}^{-n}(\mathcal{F})[n]$ (see the construction of [Spa88, 3.7], and the flabbiness result in Lemma 1.3.7). Hence f is indeed a quasi-isomorphism. □

1.4 Derived functors

In this section we construct the derived functors of $\mathrm{Hom}_{\mathcal{A}}(-,-)$ and $(- \otimes_{\mathcal{A}} -)$. Our reference for derived functors is [Del73, 1.2] (see also [Kel96, sections 13-15] for details).

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be commutative ringed spaces, and let $\mathcal{A}$ (resp. $\mathcal{B}$) be a dg-algebra on X (resp. Y). Consider a triangulated functor $F : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{B})$. Following Deligne, one says that the right derived functor RF is defined at an object $\mathcal{F} \in \mathcal{H}(\mathcal{A})$ if $\mathcal{F}$ has a right resolution $\mathcal{X}$ which is F-split[1] on the right, *i.e.* every right resolution $\mathcal{Y}$ of $\mathcal{X}$ has itself a right resolution $\mathcal{Z}$ such that F induces a quasi-isomorphism between $F(\mathcal{X})$ and $F(\mathcal{Z})$ (see [Kel96, section 14]). Similarly, left derived functors are defined at objects which are F-split on the left.

Let us remark that a K-injective $\mathcal{A}$-dg-module is F-split on the right for any such functor (this follows *e.g.* from condition (i) in definition 1.3.1). Hence, under assumptions (†), (††), right derived functors are defined on the whole category $\mathcal{D}(\mathcal{A})$, by Theorem 1.3.8.

Let Ab denote the category of abelian groups, $\mathcal{H}(\mathrm{Ab})$ its homotopy category of complexes, and $\mathcal{D}(\mathrm{Ab})$ its derived category. Let us first consider the bifunctor

$$\mathrm{Hom}_{\mathcal{A}}(-,-) : \mathcal{H}(\mathcal{A})^{\mathrm{op}} \times \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathrm{Ab}).$$

Fix an object $\mathcal{F}$ of $\mathcal{H}(\mathcal{A})^{\mathrm{op}}$. Then we define the functor $R\mathrm{Hom}_{\mathcal{A}}(\mathcal{F},-) : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathrm{Ab})$ as the right derived functor of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F},-) : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathrm{Ab})$ in the sense of Deligne. It is defined on the whole category $\mathcal{D}(\mathcal{A})$ by Theorem 1.3.8. Now for each object $\mathcal{G}$ of $\mathcal{D}(\mathcal{A})$, the functor $R\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{G}) : \mathcal{H}(\mathcal{A})^{\mathrm{op}} \to \mathcal{D}(\mathrm{Ab})$ sends quasi-isomorphisms to isomorphisms, hence factorizes to a functor $\mathcal{D}(\mathcal{A})^{\mathrm{op}} \to \mathcal{D}(\mathrm{Ab})$, again denoted $R\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{G})$. Thus, the derived bifunctor

$$R\mathrm{Hom}_{\mathcal{A}}(-,-) : \mathcal{D}(\mathcal{A})^{\mathrm{op}} \times \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathrm{Ab})$$

is well defined.

Now we consider the bifunctor

$$(- \otimes_{\mathcal{A}} -) : \mathcal{H}^r(\mathcal{A}) \times \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{O}_X).$$

As above, for each $\mathcal{F}$ in $\mathcal{H}^r(\mathcal{A})$, by Theorem 1.3.5 and Lemma 1.3.6 there are enough objects split on the left (*e.g.* K-flat dg-modules) for the functor $(\mathcal{F} \otimes_{\mathcal{A}} -) : \mathcal{H}(\mathcal{A}) \to \mathcal{H}(\mathcal{O}_X)$. Hence, its left derived functor $(\mathcal{F} \overset{L}{\otimes}_{\mathcal{A}} -) : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{O}_X)$ is well defined. And thus the derived bifunctor

$$(- \overset{L}{\otimes}_{\mathcal{A}} -) : \mathcal{D}^r(\mathcal{A}) \times \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{O}_X)$$

is well defined.

[1] Spaltenstein uses the term "unfolded", see [Spa88, p. 123].

1.5 Direct and inverse image functors

As above, let $(Y, \mathcal{O}_Y)$ be a second ringed space, and $\mathcal{B}$ a sheaf of dg-algebras on it (we call such a pair a *dg-ringed space*). A *morphism of dg-ringed spaces* $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is a morphism $f_0 : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces, together with a morphism of sheaves of dg-algebras $f_0^*\mathcal{B} \to \mathcal{A}$ (where $f_0^*\mathcal{B}$ is the usual inverse image of $\mathcal{B}$, which has a natural structure of a sheaf of dg-algebras on X).

We have a natural direct image functor

$$f_* : \mathcal{C}(X, \mathcal{A}) \to \mathcal{C}(Y, \mathcal{B})$$

and its right derived functor

$$Rf_* : \mathcal{D}(X, \mathcal{A}) \to \mathcal{D}(Y, \mathcal{B}).$$

It can be computed by means of right K-injective resolutions (see the beginning of 1.4).

Similarly, there is a natural inverse image functor

$$f^* : \begin{cases} \mathcal{C}(Y, \mathcal{B}) & \to & \mathcal{C}(X, \mathcal{A}) \\ \mathcal{F} & \mapsto & \mathcal{A} \otimes_{f_0^*\mathcal{B}} f_0^*\mathcal{F} \end{cases} .$$

Its left derived functor

$$Lf^* : \mathcal{D}(Y, \mathcal{B}) \to \mathcal{D}(X, \mathcal{A})$$

is defined on the whole of $\mathcal{D}(\mathcal{A})$, and can be computed by means of left K-flat resolutions (because f_0^* sends K-flat $\mathcal{B}$-dg-modules to K-flat $f_0^*\mathcal{B}$-dg-modules).

The following definition is adapted from [Spa88, 5.11]:

Definition 1.5.1. The $\mathcal{A}$-dg-module $\mathcal{F}$ is said to be *weakly K-injective* if $\mathrm{Hom}_{\mathcal{A}}(\mathcal{G}, \mathcal{F})$ is acyclic for any acyclic K-flat $\mathcal{A}$-dg-module $\mathcal{G}$.

It is clear from this definition that a K-injective dg-module is weakly K-injective. The following lemma is a more general (but weaker) version of Lemma 1.3.4.

Lemma 1.5.2. *Let $\mathcal{F}$ be a weakly K-injective $\mathcal{A}$-dg-module. Then $f_*\mathcal{F}$ is a weakly K-injective $\mathcal{B}$-dg-module. In particular, a weakly K-injective $\mathcal{A}$-dg-module is also weakly K-injective when considered as an $\mathcal{O}_X$-dg-module.*

Proof. Let $\mathcal{G}$ be an acyclic, K-flat $\mathcal{B}$-dg-module. By standard adjunction,

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{G}, f_*\mathcal{F}) \cong \mathrm{Hom}_{f_0^*\mathcal{B}}(f_0^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{A}}(f^*\mathcal{G}, \mathcal{F}).$$

Now $f^*\mathcal{G}$ is a K-flat $\mathcal{A}$-dg-module, and is acyclic by Lemma 1.3.6. The result follows. The second statement follows from the first one, applied to the natural morphism $(X, \mathcal{A}) \to (X, \mathcal{O}_X)$ given by the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{A}$. $\square$

Let $\mathrm{For} : \mathcal{D}(X, \mathcal{A}) \to \mathcal{D}(X, \mathcal{O}_X)$ and $\mathrm{For} : \mathcal{D}(Y, \mathcal{B}) \to \mathcal{D}(Y, \mathcal{O}_Y)$ denote the forgetful functors. Let $R(f_0)_* : \mathcal{D}(X, \mathcal{O}_X) \to \mathcal{D}(Y, \mathcal{O}_Y)$ be the right derived functor of the morphism of dg-ringed spaces $f_0 : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.

Corollary 1.5.3. (i) *The following diagram is commutative:*

$$\begin{tikzcd} \mathcal{D}(X,\mathcal{A}) \arrow[r, "Rf_*"] \arrow[d, "\mathrm{For}"'] & \mathcal{D}(Y,\mathcal{B}) \arrow[d, "\mathrm{For}"] \\ \mathcal{D}(X,\mathcal{O}_X) \arrow[r, "R(f_0)_*"] & \mathcal{D}(Y,\mathcal{O}_Y). \end{tikzcd}$$

(ii) *If $(Z,\mathcal{C})$ is a third dg-ringed space, $g:(Y,\mathcal{B})\to(Z,\mathcal{C})$ a morphism of dg-ringed spaces, the natural morphism of functors $R(g\circ f)_*\to Rg_*\circ Rf_*$ is an isomorphism.*

Proof. (i) The commutativity of the diagram is clear from the second sentence in Lemma 1.5.2, and [Spa88, 6.7] (which says, in particular, that $R(f_0)_*$ can be computed using a weakly K-injective resolution).

(ii) If $\mathcal{F}$ is a weakly K-injective $\mathcal{A}$-dg-module which is acyclic, then $\mathcal{F}$ is also acyclic and weakly K-injective as an $\mathcal{O}_X$-dg-module (by Lemma 1.5.2). Hence $f_*\mathcal{F}=(f_0)_*\mathcal{F}$ is also acyclic (see [Spa88, 5.16]). It follows that weakly K-injective dg-modules are split for direct image functors. Then the result follows from classical facts on the composition of derived functors (see [Kel96, 14.2]). □

Similarly to part (ii) of the preceding corollary, one has:

Proposition 1.5.4. *If $g:(Y,\mathcal{B})\to(Z,\mathcal{C})$ is a second morphism of dg-ringed spaces, then there exists an isomorphism of functors $L(g\circ f)^*\cong Lf^*\circ Lg^*$.*

Proof. This easily follows from the fact that g^* sends K-flat $\mathcal{C}$-dg-modules to K-flat $\mathcal{B}$-dg-modules, using again [Kel96, 14.2]. □

Definition 1.5.5. The morphism $f:(X,\mathcal{A})\to(Y,\mathcal{B})$ is a *quasi-isomorphism* if $X=Y$, $f_0=\mathrm{Id}$, and the associated morphism $\phi:\mathcal{B}\to\mathcal{A}$ induces an isomorphism on cohomology.

The following result is an immediate extension of [BL94, Theorem 10.12.5.1]. It says that the category $\mathcal{D}(X,\mathcal{A})$ depends on $\mathcal{A}$ only up to quasi-isomorphism (of course, it depends on X only up to isomorphism).

Proposition 1.5.6. *Let $f:(X,\mathcal{A})\to(X,\mathcal{B})$ be a quasi-isomorphism. Then*

$$Rf_*:\mathcal{D}(X,\mathcal{A})\to\mathcal{D}(X,\mathcal{B}) \quad \textit{and} \quad Lf^*:\mathcal{D}(X,\mathcal{B})\to\mathcal{D}(X,\mathcal{A})$$

are equivalences of categories, quasi-inverse to each other.

Proof. In our situation the functor $f_*:\mathcal{C}(X,\mathcal{A})\to\mathcal{C}(X,\mathcal{B})$ is just the restriction of scalars. In particular it takes quasi-isomorphisms to quasi-isomorphisms, hence $Rf_*:\mathcal{D}(X,\mathcal{A})\to\mathcal{D}(X,\mathcal{B})$ is also the restriction of scalars. The functor Lf^* is the derived tensor product $\mathcal{A}\overset{L}{\otimes}_{\mathcal{B}}-$. There are natural morphisms of functors $\mathrm{Id}\to Rf_*\circ Lf^*$ and $Lf^*\circ Rf_*\to\mathrm{Id}$ (these morphisms come from adjunction, as we will see in the next subsection, but we do not need it here). Let us show that they are isomorphisms.

Let $\mathcal{G}$ be a $\mathcal{B}$-dg-module, which we can assume to be K-flat. Then the morphism $\mathcal{G} \to Rf_*(Lf^*\mathcal{G}) \cong \mathcal{A} \otimes_{\mathcal{B}} \mathcal{G}$ can be represented by $\phi \otimes \mathrm{Id} : \mathcal{B} \otimes_{\mathcal{B}} \mathcal{G} \to \mathcal{A} \otimes_{\mathcal{B}} \mathcal{G}$ which is a quasi-isomorphism (because $\mathcal{G}$ is K-flat).

Let $\mathcal{F}$ be an $\mathcal{A}$-dg-module, and let $p : \mathcal{P} \to \mathcal{F}$ be a left K-flat resolution of $\mathcal{F}$ viewed as a $\mathcal{B}$-dg-module. Then the natural morphism $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{P} \cong (Lf^* \circ Rf_*)\mathcal{F} \to \mathcal{F}$ is a quasi-isomorphism, because it fits into the following commutative diagram, where the two other maps are quasi-isomorphisms:

$$\begin{tikzcd}
& \mathcal{B} \otimes_{\mathcal{B}} \mathcal{P} \cong \mathcal{P} \arrow[dl, "\phi\otimes\mathrm{Id}"'] \arrow[dr, "p"] & \\
\mathcal{A} \otimes_{\mathcal{B}} \mathcal{P} \arrow[rr] & & \mathcal{F}.
\end{tikzcd}$$

This concludes the proof. $\square$

1.6 Adjunction

Let $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a morphism of dg-ringed spaces. In this subsection we show that Rf_* and Lf^* are adjoint functors. This proof is again adapted from [Spa88].

Following [Spa88, 5.0], we denote by $\mathfrak{P}(X)$ the class of dg-modules $\mathcal{F}$ in $\mathcal{C}(X, \mathcal{O}_X)$ which are bounded above, and such that for each $i \in \mathbb{Z}$, $\mathcal{F}^i$ is a direct sum of sheaves of the form $\mathcal{O}_{U \subset X}$ (the extension by zero of $\mathcal{O}_X|_U$ to X) for U open in X. We denote[2] by $\underrightarrow{\mathfrak{P}}(X)$ the smallest full subcategory of $\mathcal{C}(X, \mathcal{O}_X)$ containing $\mathfrak{P}(X)$ and such that for any direct system $(\mathcal{F}_n)_{n \geq 0}$ of objects of $\underrightarrow{\mathfrak{P}}(X)$ such that the morphisms $\mathcal{F}_n \to \mathcal{F}_{n+1}$ are injective and split as morphisms of graded $\mathcal{A}$-modules, the object $\varinjlim \mathcal{F}_n$ is in $\underrightarrow{\mathfrak{P}}(X)$. The objects in $\underrightarrow{\mathfrak{P}}(X)$ are K-flat (as in [Spa88, 5.5]).

Lemma 1.6.1. *Let $\mathcal{F}$ be a K-flat $\mathcal{A}$-dg-module, and $\mathcal{G}$ a weakly K-injective, acyclic $\mathcal{A}$-dg-module. Then the complex of abelian groups $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ is acyclic.*

Proof. By Lemma 1.5.2, $\mathcal{G}$ is also weakly K-injective as an $\mathcal{O}_X$-dg-module. Consider the class $\mathfrak{Q}$ of objects $\mathcal{E}$ of $\mathcal{C}(X, \mathcal{A})$ such that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{G})$ is acyclic. By [Spa88, 5.20] and (1.2.2), $\mathfrak{Q}$ contains the class $\mathfrak{C}$ of objects of the form $\mathrm{Ind}(\mathcal{M})$ for $\mathcal{M} \in \underrightarrow{\mathfrak{P}}(X)$. Now, using the same proof as that of Theorem 1.3.5, there exists a direct system $(\mathcal{P}_{\leq n})_{n \geq 0}$ of $\mathcal{A}$-dg-modules such that each $\mathcal{P}_{\leq n}$ has a finite filtration which subquotients in $\mathfrak{C}$ and such that the morphisms $\mathcal{P}_{\leq n} \to \mathcal{P}_{\leq n+1}$ are injective and split as morphisms of graded $\mathcal{A}$-modules, and a quasi-isomorphism $\mathcal{P} := \varinjlim \mathcal{P}_{\leq n} \to \mathcal{F}$. Using again [Spa88, 2.3, 2.4], $\mathcal{P}$ is in $\mathfrak{Q}$. As $\mathcal{G}$ is weakly K-injective, and $\mathcal{F}$ and $\mathcal{P}$ are K-flat, the morphism $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{P}, \mathcal{G})$ is a quasi-isomorphism. The result follows. $\square$

Theorem 1.6.2. *For $\mathcal{F} \in \mathcal{D}(Y, \mathcal{B})$ and $\mathcal{G} \in \mathcal{D}(X, \mathcal{A})$, there exists a functorial isomorphism*

$$R\mathrm{Hom}_{\mathcal{A}}(Lf^*\mathcal{F}, \mathcal{G}) \cong R\mathrm{Hom}_{\mathcal{B}}(\mathcal{F}, Rf_*\mathcal{G}).$$

[2] This subcategory is a priori smaller than the one considered in [Spa88, 2.9], which allows more general direct limits, but it will be sufficient for us.

In particular, the functors Lf^* and Rf_* are adjoint.

Proof. We can assume $\mathcal{F}$ is K-flat and $\mathcal{G}$ is K-injective (by Theorems 1.3.5 and 1.3.8). Then $f_*\mathcal{G}$ is weakly K-injective by Lemma 1.5.2, and isomorphic to $Rf_*\mathcal{G}$. Hence the result follows from the classical adjunction in $\mathcal{C}(X, \mathcal{A})$ and $\mathcal{C}(Y, \mathcal{B})$ since, by Lemma 1.6.1, one can compute $R\mathrm{Hom}_{\mathcal{B}}(-,-)$ using a K-flat resolution of the first argument and a weakly K-injective resolution of the second argument. □

Remark 1.6.3. The adjunction also follows from the general result [Kel96, 13.6].

1.7 The $\mathbb{G}_{\mathbf{m}}$-equivariant case

In this subsection we show how one can adapt the preceding constructions to the case when $\mathcal{A}$ is equipped with a second grading, which we call the "internal grading". More precisely, in addition to the assumptions of 1.1, we assume we are given a decomposition $\mathcal{A} \cong \oplus_{n \in \mathbb{Z}} \mathcal{A}_n$ as an $\mathcal{O}_X$-dg-module such that, for every n, m in $\mathbb{Z}$, $\mu_{\mathcal{A}}(\mathcal{A}_n \otimes \mathcal{A}_m) \subset \mathcal{A}_{n+m}$. We call such a data a $\mathbb{G}_{\mathbf{m}}$-*equivariant dg-algebra* (in short: $\mathbb{G}_{\mathbf{m}}$-dg-algebra). Geometrically, if we equip the topological space X with a trivial $\mathbb{G}_{\mathbf{m}}$-action, such a grading indeed corresponds to a $\mathbb{G}_{\mathbf{m}}$-equivariant structure. In what follows, $\mathcal{O}_X$ will be considered as a $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebra concentrated in degree 0 for both gradings.

To avoid confusion, the first grading of $\mathcal{A}$ will be called the "cohomological grading". When a homogeneous element of $\mathcal{A}$ has cohomological degree i and internal degree j, we also say that it has bidegree (i, j).

We keep the assumptions (†) and (††) of 1.3. In particular, in this subsection all $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebras are assumed to be non-positively graded for the cohomological grading.

We define as above the notion of $\mathbb{G}_{\mathbf{m}}$-*equivariant* $\mathcal{A}$-*dg-module* (in short: $\mathbb{G}_{\mathbf{m}}$-$\mathcal{A}$-dg-module). This is a sheaf of bigraded $\mathcal{A}$-modules $\mathcal{F} = \bigoplus_{n,m \in \mathbb{Z}} \mathcal{F}^n_m$, equipped with a differential $d_{\mathcal{F}}$ of bidegree $(1, 0)$ satisfying the natural compatibility condition. In a similar way we define morphisms between dg-modules, and the categories $\mathcal{C}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{A})$, $\mathcal{H}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{A})$, $\mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{A})$. We also have natural bifunctors $\mathrm{Hom}_{\mathcal{A}, \mathbb{G}_{\mathbf{m}}}(-,-)$ and $(- \otimes_{\mathcal{A}, \mathbb{G}_{\mathbf{m}}} -)$ defined as follows. If $\mathcal{F}$, resp. $\mathcal{G}$, is a right, resp. left, $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{A}$-dg-module, then $\mathcal{F} \otimes_{\mathcal{A}, \mathbb{G}_{\mathbf{m}}} \mathcal{G}$ is isomorphic to $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$, with its natural bigrading. And if $\mathcal{F}$ and $\mathcal{G}$ are $\mathbb{G}_{\mathbf{m}}$-equivariant left $\mathcal{A}$-dg-modules, then $\mathrm{Hom}_{\mathcal{A}, \mathbb{G}_{\mathbf{m}}}(\mathcal{F}, \mathcal{G})$ is the complex of $\mathbb{Z}$-graded abelian groups whose (p, q) term consists of the morphisms of $\mathcal{A}$-modules (not necessarily commuting with the differential) mapping $\mathcal{F}^i_j$ inside $\mathcal{G}^{i+p}_{j+q}$.

We also define the notions of $\mathbb{G}_{\mathbf{m}}$-*equivariant K-injective* (respectively $\mathbb{G}_{\mathbf{m}}$-*equivariant K-flat*) $\mathcal{A}$-dg-modules, replacing the bifunctor $\mathrm{Hom}_{\mathcal{A}}(-,-)$ by $\mathrm{Hom}_{\mathcal{A}, \mathbb{G}_{\mathbf{m}}}(-,-)$ (respectively $(- \otimes_{\mathcal{A}} -)$ by $(- \otimes_{\mathcal{A}, \mathbb{G}_{\mathbf{m}}} -)$). If $\mathcal{A} = \mathcal{O}_X$, then a $\mathbb{G}_{\mathbf{m}}$-equivariant dg-module is just a direct sum of $\mathcal{O}_X$-dg-modules indexed by $\mathbb{Z}$.

Lemma 1.7.1. *A $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module $\mathcal{G}$ is $\mathbb{G}_{\mathbf{m}}$-equivariant K-injective (resp. K-flat) if and only if each of its internal graded components $\mathcal{G}_m$ is K-injective (resp. K-flat).*

Proof. We only give a proof for the K-injective case (the K-flat case is similar and easier). Let $\mathcal{F}$ be another $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module. Then $\mathrm{Hom}_{\mathcal{O}_X,\mathbb{G}_{\mathbf{m}}}(\mathcal{F},\mathcal{G})$ is a complex of graded abelian groups. It is exact if and only if each of its graded components is. Hence for any $m \in \mathbb{Z}$ we have to consider the complex with n-th component $\prod_{i,j} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_X)}(\mathcal{F}_j^i, \mathcal{G}_{j+m}^{i+n})$. This complex is the product (for $j \in \mathbb{Z}$) of the complexes with n-th component $\prod_i \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_X)}(\mathcal{F}_j^i, \mathcal{G}_{j+m}^{i+n})$, *i.e.* $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_j^\bullet, \mathcal{G}_{j+m}^\bullet)$. As the product is exact on abelian groups, our complex $\mathrm{Hom}_{\mathcal{O}_X,\mathbb{G}_{\mathbf{m}}}(\mathcal{F},\mathcal{G})$ is exact if and only if for any m and j in $\mathbb{Z}$ the complex $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_j^\bullet, \mathcal{G}_{j+m}^\bullet)$ is exact. The result follows. □

It follows from this lemma that there are enough K-injective and K-flat objects in $\mathcal{C}_{\mathbb{G}_{\mathbf{m}}}(X,\mathcal{O}_X)$. Then the proofs of Theorems 1.3.5 and 1.3.8 generalize, thus there are enough K-injective and K-flat objects in $\mathcal{C}_{\mathbb{G}_{\mathbf{m}}}(X,\mathcal{A})$ for any $\mathcal{A}$ (to generalize these proofs, one has to replace the induction and coinduction functors by $\mathbb{G}_{\mathbf{m}}$-equivariant analogues). Hence one can construct the derived bifunctors $R\mathrm{Hom}_{\mathcal{A},\mathbb{G}_{\mathbf{m}}}(-,-)$ and $(- \overset{L}{\otimes}_{\mathcal{A},\mathbb{G}_{\mathbf{m}}} -)$.

Let $\mathrm{For} : \mathcal{C}_{\mathbb{G}_{\mathbf{m}}}(X,\mathcal{A}) \to \mathcal{C}(X,\mathcal{A})$ denote the forgetful functor, sending $\mathcal{F}$ to the dg-module with n-th component $\bigoplus_{m\in\mathbb{Z}} \mathcal{F}_m^n$.

Lemma 1.7.2. *For every $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{A}$-dg-module $\mathcal{F}$, there exists a $\mathbb{G}_{\mathbf{m}}$-equivariant K-flat $\mathcal{A}$-dg-module $\mathcal{P}$ and a $\mathbb{G}_{\mathbf{m}}$-equivariant quasi-isomorphism $\mathcal{P} \to \mathcal{F}$ such that the image $\mathrm{For}(\mathcal{P}) \to \mathrm{For}(\mathcal{F})$ is a K-flat resolution in $\mathcal{C}(X,\mathcal{A})$.*

Proof. This lemma follows from the fact that for the dg-algebra $\mathcal{O}_X$, the image under For of a $\mathbb{G}_{\mathbf{m}}$-equivariant K-flat dg-module is a K-flat dg-module (by Lemma 1.7.1 and the fact that a direct sum of K-flat modules is K-flat), and the construction of a resolution given in the proof of Theorem 1.3.5, which is parallel for the $\mathbb{G}_{\mathbf{m}}$-equivariant and the non $\mathbb{G}_{\mathbf{m}}$-equivariant case. □

It follows from this lemma that the bifunctors $(- \overset{L}{\otimes}_{\mathcal{A},\mathbb{G}_{\mathbf{m}}} -)$ and $(- \overset{L}{\otimes}_{\mathcal{A}} -)$ correspond under the forgetful functors. Hence from now on we will denote both bifunctors by $(- \overset{L}{\otimes}_{\mathcal{A}} -)$.

Now we consider direct and inverse image functors. Let $(Y,\mathcal{B})$ be a second $\mathbb{G}_{\mathbf{m}}$-equivariant dg-ringed space, and $f : (X,\mathcal{A}) \to (Y,\mathcal{B})$ a $\mathbb{G}_{\mathbf{m}}$-equivariant morphism of dg-ringed spaces. There are natural functors

$$(f_{\mathbb{G}_{\mathbf{m}}})_* : \mathcal{C}_{\mathbb{G}_{\mathbf{m}}}(X,\mathcal{A}) \to \mathcal{C}_{\mathbb{G}_{\mathbf{m}}}(Y,\mathcal{B}) \quad \text{and} \quad (f_{\mathbb{G}_{\mathbf{m}}})^* : \mathcal{C}_{\mathbb{G}_{\mathbf{m}}}(Y,\mathcal{B}) \to \mathcal{C}_{\mathbb{G}_{\mathbf{m}}}(X,\mathcal{A})$$

and their derived functors

$$R(f_{\mathbb{G}_{\mathbf{m}}})_* : \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(X,\mathcal{A}) \to \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(Y,\mathcal{B}) \quad \text{and} \quad L(f_{\mathbb{G}_{\mathbf{m}}})^* : \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(Y,\mathcal{B}) \to \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(X,\mathcal{A}).$$

These functors are adjoint (the same proof as in the non $\mathbb{G}_{\mathbf{m}}$-equivariant case works). It follows from Lemma 1.7.2 that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(Y,\mathcal{B}) \arrow[r, "L(f_{\mathbb{G}_{\mathbf{m}}})^*"] \arrow[d, "\mathrm{For}"] & \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(X,\mathcal{A}) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}(Y,\mathcal{B}) \arrow[r, "Lf^*"] & \mathcal{D}(X,\mathcal{A}).
\end{tikzcd}$$

In order to prove the similar result for $R(f_{\mathbb{G}_{\mathbf{m}}})_*$, we need some preparation.

First, consider the case of the dg-algebra $\mathcal{O}_X$. Recall the notation of 1.6.

Definition 1.7.3. $\mathcal{F} \in \mathcal{C}(X, \mathcal{O}_X)$ is said to be *K-limp* if $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is acyclic for every acyclic complex $\mathcal{S}$ in $\mathfrak{P}(X)$.

Note that this notion (also considered in [Spa88, 5.11]) is weaker than weak K-injectivity.

As X is assumed to be noetherian, a direct sum of flabby sheaves on X is flabby ([Har77, III.2.8] or [God64, 3.10]). Moreover, for every open $U \subset X$ the functor $\Gamma(U, -)$ commutes with infinite direct sums ([Har77, III.2.9] or [God64, 3.10.1]). If $\mathcal{F}$ is a bounded below $\mathcal{O}_X$-dg-module, $R\Gamma(U, \mathcal{F})$ can be computed using a flabby resolution. Hence the functor $R\Gamma(U, -)$ commutes with infinite direct sums in the case of a family of $\mathcal{O}_X$-dg-modules which are uniformly bounded below. Now we will generalize this fact.

Lemma 1.7.4. *A direct sum of K-limp $\mathcal{O}_X$-dg-modules is K-limp.*

Proof. Let $(\mathcal{F}_j)_{j\in J}$ be K-limp $\mathcal{O}_X$-dg-modules. Let $\bigoplus_{j\in J} \mathcal{F}_j \to \mathcal{I}$ be a K-injective resolution, constructed as in [Spa88, 3.7, 3.13]. Using [Spa88, 5.17], it will be sufficient to prove that for every open $U \subset X$, the morphism $\Gamma(U, \bigoplus_{j\in J} \mathcal{F}_j) = \bigoplus_{j\in J} \Gamma(U, \mathcal{F}_j) \to \Gamma(U, \mathcal{I})$ is a quasi-isomorphism. We fix an open U, and $m \in \mathbb{Z}$. We have $\mathcal{I} \cong \varprojlim_n \mathcal{I}_n$ where $\mathcal{I}_n$ is a K-injective resolution of $\tau_{\geq -n}(\bigoplus_{j\in J} \mathcal{F}_j) \cong \bigoplus_{j\in J} \tau_{\geq -n}\mathcal{F}_j$. Then for N sufficiently large, we have an isomorphism $H^m(\Gamma(U, \mathcal{I})) \cong H^m(\Gamma(U, \mathcal{I}_N))$ (see the proof of [Spa88, 3.13]). But $H^m(\Gamma(U, \mathcal{I}_N)) \cong R^m\Gamma(U, \bigoplus_{j\in J} \tau_{\geq -N}\mathcal{F}_j)$. Using the remark before the lemma, the latter is isomorphic to $\bigoplus_{j\in J} R^m\Gamma(U, \tau_{\geq -N}\mathcal{F}_j)$. For the same reason, for N sufficiently large (uniformly in j) we have $R^m\Gamma(U, \tau_{\geq -N}\mathcal{F}_j) \cong R^m\Gamma(U, \mathcal{F}_j)$. We conclude using the fact that, as $\mathcal{F}_j$ is K-limp, by [Spa88, 6.4] we have $R^m\Gamma(U, \mathcal{F}_j) \cong H^m(\Gamma(U, \mathcal{F}_j))$. □

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. We may also consider it as a morphism of $\mathbb{G}_{\mathbf{m}}$-equivariant dg-ringed spaces (with trivial $\mathbb{G}_{\mathbf{m}}$-action on $\mathcal{O}_X$ and $\mathcal{O}_Y$).

Corollary 1.7.5. *For every family of objects $(\mathcal{F}_i)_{i\in I}$ of $\mathcal{C}(X, \mathcal{O}_X)$ we have $Rf_*(\bigoplus_{i\in I} \mathcal{F}_i) \cong \bigoplus_{i\in I} Rf_*(\mathcal{F}_i)$. In particular, the following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{O}_X) \arrow[r, "R(f_{\mathbb{G}_{\mathbf{m}}})_*"] \arrow[d, "\mathrm{For}"] & \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(Y, \mathcal{O}_Y) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}(X, \mathcal{O}_X) \arrow[r, "Rf_*"] & \mathcal{D}(Y, \mathcal{O}_Y).
\end{tikzcd}$$

Proof. The isomorphism follows from the facts that f_* commutes with direct sums (because X is noetherian), that Rf_* can be computed by means of K-limp resolutions ([Spa88, 6.7]), and Lemma 1.7.4.

Then the commutativity of the diagram follows from this isomorphism and the obvious isomorphism $\mathrm{For} \circ R(f_{\mathbb{G}_{\mathbf{m}}})_*(\mathcal{F}) \cong \bigoplus_{n\in\mathbb{Z}} Rf_*(\mathcal{F}_n)$ for a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module $\mathcal{F}$ with decomposition $\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}_n$. □

Let $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a morphism of $\mathbb{G}_{\mathbf{m}}$-equivariant dg-ringed spaces.

Corollary 1.7.6. *The following diagrams are commutative:*

$$\begin{tikzcd} \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{A}) \arrow[r, "R(f_{\mathbb{G}_{\mathbf{m}}})_*"] \arrow[d, "\mathrm{For}"] & \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(Y, \mathcal{B}) \arrow[d, "\mathrm{For}"] \\ \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{O}_X) \arrow[r, "R(f_{0,\mathbb{G}_{\mathbf{m}}})_*"] & \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(Y, \mathcal{O}_Y), \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{A}) \arrow[r, "R(f_{\mathbb{G}_{\mathbf{m}}})_*"] \arrow[d, "\mathrm{For}"] & \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(Y, \mathcal{B}) \arrow[d, "\mathrm{For}"] \\ \mathcal{D}(X, \mathcal{A}) \arrow[r, "Rf_*"] & \mathcal{D}(Y, \mathcal{B}). \end{tikzcd}$$

Proof. The commutativity of the second diagram follows from the commutativity of the first one and corollaries 1.5.3 and 1.7.5. Hence we only have to prove that the first diagram is commutative. Now consider a $\mathbb{G}_{\mathbf{m}}$-equivariant K-injective $\mathcal{A}$-dg-module $\mathcal{F}$. By an analogue of Lemma 1.5.2, $\mathcal{F}$ is weakly K-injective as a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module. Hence each of its graded components is weakly K-injective as an $\mathcal{O}_X$-dg-module (see the proof of Lemma 1.7.1). The result follows, since one can compute $R(f_{0,\mathbb{G}_{\mathbf{m}}})_*$ using K-limp resolutions of each components. □

Proofs similar to those of subsection 1.5 show that if $g : (Y, \mathcal{B}) \to (Z, \mathcal{C})$ is a second morphism of $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebras, one has isomorphisms

$$R((g \circ f)_{\mathbb{G}_{\mathbf{m}}})_* \cong R(g_{\mathbb{G}_{\mathbf{m}}})_* \circ R(f_{\mathbb{G}_{\mathbf{m}}})_*, \tag{1.7.7}$$

$$L((g \circ f)_{\mathbb{G}_{\mathbf{m}}})^* \cong L(f_{\mathbb{G}_{\mathbf{m}}})^* \circ L(g_{\mathbb{G}_{\mathbf{m}}})^*. \tag{1.7.8}$$

Remark 1.7.9. One of the motivations for introducing $\mathbb{G}_{\mathbf{m}}$-equivariant dg-modules comes from the following situation, that we will encounter later in section 2. Let X be a variety, and $\mathcal{F}$ a locally free $\mathcal{O}_X$-module. Consider the dg-algebra $\mathcal{A} = S_{\mathcal{O}_X}(\mathcal{F})$, the symmetric algebra of $\mathcal{F}$ over $\mathcal{O}_X$, with trivial differential and the grading such that $\mathcal{F}$ is in degree 2. This dg-algebra is not concentrated in non-positive degrees, hence we cannot apply the constructions of subsections 1.3 to 1.6. Now, let us consider $\mathcal{A}$ as a $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebra, with $\mathcal{F}$ in bidegree $(2, -2)$. Let $\mathcal{B}$ denote the $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebra which is also isomorphic to $S_{\mathcal{O}_X}(\mathcal{F})$ as a sheaf of algebras, with trivial differential, and with $\mathcal{F}$ in bidegree $(0, -2)$. Then the "regrading" functor

$$\xi : \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(\mathcal{A}) \to \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(\mathcal{B})$$

defined by $\xi(\mathcal{M})^i_j := \mathcal{M}^{i-j}_j$ is an equivalence of categories. Using this equivalence and the fact that $\mathcal{B}$ is concentrated in non-positive degrees, all the constructions and results obtained in 1.7 can be transfered to the $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebra $\mathcal{A}$.

1.8 Dg-schemes and dg-sheaves

In this section we define dg-schemes and dg-sheaves over them. Our reference is [CFK01, section 2], but we modify some definitions according to our purposes.

Definition 1.8.1. A *dg-scheme* is a dg-ringed space $X = (X^0, \mathcal{O}_X^\bullet)$ where X^0 is an ordinary scheme and $\mathcal{O}_X^\bullet$ is a sheaf of non-positively graded, graded-commutative dg-algebras on X^0, such that each $\mathcal{O}_X^i$ is a quasi-coherent $\mathcal{O}_{X^0}$-module (the structure of $\mathcal{O}_{X^0}$-module being given by the action of the image of $\mathcal{O}_{X^0}$ inside $\mathcal{O}_X^0$).

A *morphism of dg-schemes* $f : X \to Y$ is a morphism of dg-ringed spaces $f : (X, \mathcal{O}_X^\bullet) \to (Y, \mathcal{O}_Y^\bullet)$ (see 1.5).

Let us fix a dg-scheme X.

Definition 1.8.2. (i) A *quasi-coherent dg-sheaf* on X is a sheaf $\mathcal{F}$ of $\mathcal{O}_X^\bullet$-dg-modules on X^0 such that each $H^i(\mathcal{F})$ is a quasi-coherent $\mathcal{O}_{X^0}$-module. We denote by DGSh(X) the full subcategory[3] of $\mathcal{D}(X, \mathcal{O}_X^\bullet)$ whose objects are quasi-coherent dg-sheaves.

(ii) A *coherent dg-sheaf* on X is a quasi-coherent dg-sheaf $\mathcal{F}$ on X whose cohomology $H(\mathcal{F})$ is locally finitely generated over the sheaf of algebras $H(\mathcal{O}_X^\bullet)$. We denote by $\mathrm{DGCoh}(X)$ the full subcategory of $\mathcal{D}(X, \mathcal{O}_X^\bullet)$ whose objects are coherent dg-sheaves.

Remark 1.8.3. (i) If X is an ordinary scheme (*i.e.* if $\mathcal{O}_X^0 = \mathcal{O}_{X^0}$ and $\mathcal{O}_X^i = 0$ for $i \neq 0$) which is quasi-compact and separated, then the category DGSh(X) is equivalent to the (unbounded) derived category of the abelian category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X (see [BN93, 5.5]). If moreover X is noetherian, then the category $\mathrm{DGCoh}(X)$ is equivalent to the bounded derived category of the abelian category $\mathrm{Coh}(X)$ of coherent sheaves on X (see [BGI71, II.2.2.2.1]; see also [Bor87, VI.2.B] for a sketch of a more elementary proof, following J. Bernstein and P. Deligne).

(ii) If $f : X \to Y$ is a morphism of dg-schemes, then it induces functors $Rf_* : \mathcal{D}(X^0, \mathcal{O}_X^\bullet) \to \mathcal{D}(Y^0, \mathcal{O}_Y^\bullet)$ and $Lf^* : \mathcal{D}(Y^0, \mathcal{O}_Y^\bullet) \to \mathcal{D}(X^0, \mathcal{O}_X^\bullet)$. It is not clear in general if these functors restrict to functors between DGSh(X) and DGSh(Y), or between $\mathrm{DGCoh}(X)$ and $\mathrm{DGCoh}(Y)$. In practice, it will always be the case in this chapter. We will prove it in each particular case.

The following lemma follows immediately from Corollary 1.5.3 and Proposition 1.5.4 (see also Proposition 1.5.6).

Lemma 1.8.4. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of dg-schemes, with f a quasi-isomorphism (then $X^0 = Y^0$, and $f_0 = \mathrm{Id}$). The following diagrams are commutative.*

$$\begin{tikzcd}
\mathcal{D}(X, \mathcal{O}_X^\bullet) \arrow[rr, "Rf_*", "\sim"'] \arrow[dr, "R(g\circ f)_*"'] & & \mathcal{D}(Y, \mathcal{O}_Y^\bullet) \arrow[dl, "Rg_*"] \\
& \mathcal{D}(Z, \mathcal{O}_Z^\bullet), &
\end{tikzcd}$$

[3]It is not clear from this definition that this subcategory is a *triangulated* subcategory. In fact it turns out that it is the case under some reasonable conditions. In this chapter we essentially consider *coherent* dg-sheaves over bounded dg-algebras, hence this point will not be a problem.

$$\begin{tikzcd} \mathcal{D}(Y, \mathcal{O}_Y^\bullet) \arrow[rr, "Lf^*", "\sim"'] \arrow[dr, "Lg^*"'] & & \mathcal{D}(X, \mathcal{O}_X^\bullet) \arrow[dl, "L(g\circ f)^*"] \\ & \mathcal{D}(Z, \mathcal{O}_Z^\bullet). & \end{tikzcd}$$

In particular, if the functors Rg_* and Lg^* restrict to functors between the categories DGSh(Y) *and* DGSh(Z) *(or between* $\mathrm{DGCoh}(Y)$ *and* $\mathrm{DGCoh}(Z)$*), then the functors $R(g \circ f)_*$ and $L(g \circ f)^*$ also restrict to functors between* DGSh(X) *and* DGSh(Z) *(or* $\mathrm{DGCoh}(X)$ *and* $\mathrm{DGCoh}(Z)$*), and conversely.*

This result allows one to replace a given dg-scheme by a quasi-isomorphic one when convenient. Of course, given $g : Y \to Z$ we may as well replace Z by a quasi-isomorphic dg-scheme Z'. Hence we will consider dg-schemes only up to quasi-isomorphism.

As a typical example, we define the derived intersection of two closed subschemes. Consider a scheme X, and two closed subschemes Y and Z. Let us denote by $i : Y \to X$ and $j : Z \to X$ the closed embeddings. Consider the sheaf of dg-algebras $i_*\mathcal{O}_Y \overset{L}{\otimes}_{\mathcal{O}_X} j_*\mathcal{O}_Z$ on X. It is defined up to quasi-isomorphism: if $\mathcal{A}_Y \to i_*\mathcal{O}_Y$, respectively $\mathcal{A}_Z \to j_*\mathcal{O}_Z$ are quasi-isomorphisms of non-positively graded, graded-commutative sheaves of dg-algebras on X, with $\mathcal{A}_Y$ and $\mathcal{A}_Z$ quasi-coherent and K-flat over $\mathcal{O}_X$, then $i_*\mathcal{O}_Y \overset{L}{\otimes}_{\mathcal{O}_X} j_*\mathcal{O}_Z$ is quasi-isomorphic to $\mathcal{A}_Y \otimes_{\mathcal{O}_X} j_*\mathcal{O}_Z$, or to $i_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{A}_Z$, or to $\mathcal{A}_Y \otimes_{\mathcal{O}_X} \mathcal{A}_Z$.

Definition 1.8.5. The right derived intersection of Y and Z in X is the dg-scheme

$$Y \overset{R}{\cap}_X Z := (X,\, i_*\mathcal{O}_Y \overset{L}{\otimes}_{\mathcal{O}_X} j_*\mathcal{O}_Z),$$

defined up to quasi-isomorphism.

Remark 1.8.6. Let us keep the notation as above. The sheaf of dg-algebras $\mathcal{A}_Y \otimes_{\mathcal{O}_X} j_*\mathcal{O}_Z$ is isomorphic to the sheaf of dg-algebras $j_*(j^*\mathcal{A}_Y)$. Hence the direct image functor $j_* : \mathcal{C}(Z, j^*\mathcal{A}_Y) \to \mathcal{C}(X, \mathcal{A}_Y \otimes_{\mathcal{O}_X} j_*\mathcal{O}_Z)$ is an equivalence of categories. As a consequence, by abuse of notation we will often identify the dg-schemes $(Z, j^*\mathcal{A}_Y)$ and $Y \overset{R}{\cap}_X Z$.

2 Linear Koszul duality

Usual Koszul duality (see *e.g.* [BGG78], [BGS96], [GKM93]) relates modules over the symmetric algebra $S(V)$ of a finite dimensional vector space V to modules over the exterior algebra $\Lambda(V^*)$ of the dual vector space. In this section we give a relative version of this duality, and a geometric interpretation in terms of derived intersections (due to I. Mirković).

2.1 Reminder on Koszul duality

We fix a scheme $(X, \mathcal{O}_X)$. Let $\mathcal{F}$ be a locally free sheaf of finite rank over X. We denote by

$$\mathcal{S} := S_{\mathcal{O}_X}(\mathcal{F}^\vee)$$

the symmetric algebra of $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ over $\mathcal{O}_X$. We consider it as a sheaf of dg-algebras with trivial differential, and with the grading such that $\mathcal{F}^\vee$ is in degree 2. Similarly, we denote by

$$\mathcal{T} := \Lambda_{\mathcal{O}_X}(\mathcal{F})$$

the exterior algebra of $\mathcal{F}$ over $\mathcal{O}_X$, considered as a sheaf of dg-algebras with trivial differential, and the grading such that $\mathcal{F}$ is in degree -1. For the categories of dg-modules over these dg-algebras, we use the notation of section 1.

Let $\mathcal{C}^+(\mathcal{S})$ be the full subcategory of $\mathcal{C}(\mathcal{S})$ whose objects are bounded below $\mathcal{S}$-dg-modules. We define similarly $\mathcal{C}^+(\mathcal{T})$. We denote by $\mathcal{H}^+(\mathcal{S})$, $\mathcal{H}^+(\mathcal{T})$, $\mathcal{D}^+(\mathcal{S})$ and $\mathcal{D}^+(\mathcal{T})$ the homotopy and derived categories obtained by the usual procedures (see section 1).

Following [GKM93], we define the functor

$$\mathscr{A} : \mathcal{C}^+(\mathcal{S}) \to \mathcal{C}^+(\mathcal{T})$$

by setting $\mathscr{A}(\mathcal{M}) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{T}, \mathcal{M}) \cong \mathcal{T}^\vee \otimes_{\mathcal{O}_X} \mathcal{M}$, where the $\mathcal{T}$-module structure is given by the formula

$$(t \cdot \phi)(s) = (-1)^{\deg(t)(\deg(t)+1)/2} \phi(ts)$$

and the differential is defined as the sum of d_1 and d_2, where

$$d_1(\phi)(t) = (-1)^{\deg(t)} d_M(\phi(t)), \tag{2.1.1}$$

and d_2 is defined as follows. Consider the canonical morphism $\mathcal{O}_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \cong \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee$. Then d_2 is the opposite of the composition

$$\mathcal{T}^\vee \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{T}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\beta \otimes \alpha_{\mathcal{F}}} \mathcal{T}^\vee \otimes_{\mathcal{O}_X} \mathcal{M}$$

where $\alpha_{\mathcal{F}}$ is the given action $\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}$ and β is the (right) action of $\mathcal{F}$ on $\mathcal{T}^\vee$ which is the transpose of left multiplication. If t is a local section of $\mathcal{T}$ in a neighborhood of x, with $\{y_i, i \in I\}$ a basis of $\mathcal{F}_x$ as $\mathcal{O}_{X,x}$-module and $\{y_i^*, i \in I\}$ the dual basis of $(\mathcal{F}^\vee)_x$, we have

$$d_2(\phi)(t) = -\sum_i y_i^* \phi(y_i t). \tag{2.1.2}$$

Using formulas (2.1.1) and (2.1.2), one easily verifies that $d_1 + d_2$ is a differential, and that $\mathscr{A}(\mathcal{M})$ is a $\mathcal{T}$-dg-module.

We also define the functor

$$\mathscr{B} : \mathcal{C}^+(\mathcal{T}) \to \mathcal{C}^+(\mathcal{S})$$

by setting $\mathscr{B}(\mathcal{N}) := \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{N}$, where the $\mathcal{S}$-module structure is by left multiplication on $\mathcal{S}$ and the differential is the sum $d_3 + d_4$, where

$$d_3(s \otimes n) = s \otimes d_{\mathcal{N}}(n) \tag{2.1.3}$$

and d_4 is the composition $\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{N} \to \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{N} \to \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{N}$. With the same notation as above, we have

$$d_4(s \otimes m) = \sum_i s y_i^* \otimes y_i n. \tag{2.1.4}$$

Using formulas (2.1.3) and (2.1.4), one again verifies that $d_3 + d_4$ is a differential, and that $\mathscr{B}(\mathcal{N})$ is a $\mathcal{S}$-dg-module.

Taking the stalks at a point and using spectral sequence arguments (see [GKM93, 9.1]), one proves that $\mathscr{A}$ and $\mathscr{B}$ send quasi-isomorphisms to quasi-isomorphisms, and hence define functors

$$\mathscr{A} : \mathcal{D}^{+}(\mathcal{S}) \to \mathcal{D}^{+}(\mathcal{T}) \quad \text{and} \quad \mathscr{B} : \mathcal{D}^{+}(\mathcal{T}) \to \mathcal{D}^{+}(\mathcal{S}).$$

Theorem 2.1.5. *The functors $\mathscr{A}$ and $\mathscr{B}$ are equivalences of categories between $\mathcal{D}^{+}(\mathcal{S})$ and $\mathcal{D}^{+}(\mathcal{T})$, quasi-inverse to each other.*

To prove this theorem, one constructs morphisms of functors $\mathrm{Id} \to \mathscr{A} \circ \mathscr{B}$ and $\mathscr{B} \circ \mathscr{A} \to \mathrm{Id}$ as in [GKM93, section 16]. To prove that they are isomorphisms, it suffices to look at the stalks at a point x. Then the same proof as that of [GKM93] works.

2.2 Restriction to certain subcategories

Now we assume that X is a non-singular algebraic variety over an algebraically closed field $\Bbbk$. If $\mathcal{A}$ is a dg-algebra on X, we denote by $\mathcal{D}^{\mathrm{qc}}(\mathcal{A})$, resp. $\mathcal{D}^{\mathrm{qc,fg}}(\mathcal{A})$ the full subcategory of $\mathcal{D}(\mathcal{A})$ consisting of dg-modules whose cohomology is quasi-coherent as an $\mathcal{O}_X$-module, resp. whose total cohomology is quasi-coherent over $\mathcal{O}_X$ and locally finitely generated over the sheaf of algebras $H(\mathcal{A})$. Similarly we define $\mathcal{D}^{+,\mathrm{qc}}(\mathcal{A})$, $\mathcal{D}^{+,\mathrm{qc,fg}}(\mathcal{A})$, and bigraded analogues. Let $\mathcal{F}$, $\mathcal{S}$ and $\mathcal{T}$ be as in 2.1.

Lemma 2.2.1. *The equivalences $\mathscr{A}$ and $\mathscr{B}$ restrict to equivalences between $\mathcal{D}^{+,\mathrm{qc}}(\mathcal{S})$ and $\mathcal{D}^{+,\mathrm{qc}}(\mathcal{T})$.*

Proof. We only have to prove that $\mathscr{A}$ and $\mathscr{B}$ map these subcategories one into each other. But this is clear from the existence of the spectral sequences (of sheaves) analogous to the ones of [GKM93, 9.1]. □

Lemma 2.2.2. *The equivalences $\mathscr{A}$ and $\mathscr{B}$ restrict further to equivalences between the categories $\mathcal{D}^{+,\mathrm{qc,fg}}(\mathcal{S})$ and $\mathcal{D}^{+,\mathrm{qc,fg}}(\mathcal{T})$.*

Proof. We only have to prove that $\mathscr{A}$ maps $\mathcal{D}^{+,\mathrm{qc,fg}}(\mathcal{S})$ into $\mathcal{D}^{+,\mathrm{qc,fg}}(\mathcal{T})$, and that $\mathscr{B}$ maps $\mathcal{D}^{+,\mathrm{qc,fg}}(\mathcal{T})$ into $\mathcal{D}^{+,\mathrm{qc,fg}}(\mathcal{S})$. Let us first consider $\mathscr{B}$. Let $\mathcal{M}$ be an object of $\mathcal{D}^{+,\mathrm{qc,fg}}(\mathcal{T})$. By the previous lemma, $\mathscr{B}(\mathcal{M}) \in \mathcal{D}^{+,\mathrm{qc}}(\mathcal{S})$, and we have to prove that for any $x \in X$, the $\mathcal{S}_x$-dg-module $\mathscr{B}(\mathcal{M})_x$ has finitely generated cohomology. But $H(\mathcal{M}_x)$ is finitely generated over $\mathcal{O}_{X,x}$ (because it is finitely generated over $\mathcal{T}_x$, which is itself finitely generated as an $\mathcal{O}_{X,x}$-module). Thus, the E_1-term of the spectral sequence analogous to [GKM93, 9.1.4] is finitely generated over $\mathcal{S}_x$. The result follows since $\mathcal{S}_x$ is a noetherian ring.

Concerning $\mathscr{A}$, again taking stalks, one can use the arguments of [GKM93, 16.7] (since X is non-singular, $\mathcal{O}_{X,x}$ has finite homological dimension, which allows to generalize the proof). □

The inclusion $\mathcal{C}^+(\mathcal{T}) \to \mathcal{C}(\mathcal{T})$ induces a functor $\mathcal{D}^{+,\mathrm{qc},\mathrm{fg}}(\mathcal{T}) \to \mathcal{D}^{\mathrm{qc},\mathrm{fg}}(\mathcal{T})$. If $n \in \mathbb{Z}$ and $\mathcal{M}$ is a $\mathcal{T}$-dg-module, we denote by $\tau_{\geq n}\mathcal{M}$ the $\mathcal{T}$-dg-module given by

$$\cdots \to 0 \to \mathcal{M}^n/\mathrm{Im}(d^{n-1}) \to \mathcal{M}^{n+1} \to \cdots$$

Observe that this is meaningful because $\mathcal{T}$ is concentrated in non-positive degrees.

Lemma 2.2.3. *The natural functor $\mathcal{D}^{+,\mathrm{qc},\mathrm{fg}}(\mathcal{T}) \to \mathcal{D}^{\mathrm{qc},\mathrm{fg}}(\mathcal{T})$ is an equivalence of categories.*

Proof. We only have to prove that for every $\mathcal{T}$-dg-module $\mathcal{N}$ whose cohomology is locally finitely generated, there exists a bounded below $\mathcal{T}$-dg-module $\mathcal{N}'$ and a quasi-isomorphism $\mathcal{N} \xrightarrow{\mathrm{qis}} \mathcal{N}'$. Now the cohomology of $\mathcal{N}$ is bounded. If $H^i(\mathcal{N}) = 0$ for $i < n$, we may take $\mathcal{N}' = \tau_{\geq n}\mathcal{N}$. □

Remark 2.2.4. We cannot use such an argument for $\mathcal{S}$, and we do not know if the natural functor $\mathcal{D}^{+,\mathrm{qc},\mathrm{fg}}(\mathcal{S}) \to \mathcal{D}^{\mathrm{qc},\mathrm{fg}}(\mathcal{S})$ is an equivalence[4].

Combining Lemmas 2.2.2 and 2.2.3, one obtains an equivalence of categories

$$\mathcal{D}^{+,\mathrm{qc},\mathrm{fg}}(X, \mathcal{S}) \cong \mathcal{D}^{\mathrm{qc},\mathrm{fg}}(X, \mathcal{T}). \tag{2.2.5}$$

Now we give a geometric interpretation of this equivalence.

2.3 Linear Koszul Duality

We consider the following situation: E is a vector bundle over X (of finite rank), and $F \subset E$ is a sub-bundle. We denote by $p : E \to X$ the natural projection. Let $\mathcal{E}$ and $\mathcal{F}$ be the sheaves of sections of E and F (these are locally free $\mathcal{O}_X$-modules of finite rank). Let E^* be the vector bundle dual to E, let $F^\perp \subset E^*$ be the orthogonal of F (a sub-bundle of E^*), and let $q : E^* \to X$ be the projection. We define an action of $\mathbb{G}_{\mathbf{m}}$ on E and F, letting $t \in \Bbbk^\times$ act by multiplication by t^2 on the fibers. This induces an action on E^* and $F^\perp$, where $t \in \Bbbk^\times$ acts by multiplication by t^{-2} on the fibers. Now, until the end of this section, we denote by $\mathcal{S}$ and $\mathcal{T}$ the following $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebras with trivial differentials:

$$\begin{aligned} \mathcal{S} &:= S_{\mathcal{O}_X}(\mathcal{F}^\vee) \quad \text{with } \mathcal{F}^\vee \text{ in bidegree } (2,-2) \\ \mathcal{T} &:= \Lambda_{\mathcal{O}_X}(\mathcal{F}) \quad \text{with } \mathcal{F} \text{ in bidegree } (-1,2). \end{aligned}$$

Then, first, bigraded analogues of the previous constructions (see in particular (2.2.5)) yield an equivalence of categories

$$\mathcal{D}^{+,\mathrm{qc},\mathrm{fg}}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{S}) \cong \mathcal{D}^{\mathrm{qc},\mathrm{fg}}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}), \tag{2.3.1}$$

where $\mathcal{D}^+_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{S})$ is the localization with respect to quasi-isomorphisms of the homotopy category of the category $\mathcal{C}^+_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{S})$ of $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{S}$-dg-modules which are bounded below for the cohomological degree (uniformly in the internal degree), and $\mathcal{D}^{+,\mathrm{qc},\mathrm{fg}}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{S})$, $\mathcal{D}^{\mathrm{qc},\mathrm{fg}}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T})$ are defined as in 2.2.

[4] One easily sees that this is the case if *e.g.* $X = \mathrm{Spec}(\Bbbk)$.

Lemma 2.3.2. *There exists a natural equivalence of categories*

$$\mathcal{D}^b\mathrm{Coh}(E) \cong \mathcal{D}^{\mathrm{qc,fg}}(X, S_{\mathcal{O}_X}(\mathcal{E}^\vee)), \tag{2.3.3}$$

where $S_{\mathcal{O}_X}(\mathcal{E}^\vee)$ is considered as a dg-algebra concentrated in degree 0, with trivial differential.

Similarly, if $S_{\mathcal{O}_X}(\mathcal{E}^\vee)$ is regarded as a $\mathbb{G}_\mathbf{m}$-equivariant dg-algebra with trivial differential and $\mathcal{E}^\vee$ in bidegree $(0,-2)$, there exists a natural equivalence of categories

$$\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(E) \cong \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, S_{\mathcal{O}_X}(\mathcal{E}^\vee)). \tag{2.3.4}$$

Similar results hold for F, E^*, $F^\perp$.

Proof. We only give the proof of (2.3.3), the proof of (2.3.4) being similar. We denote by $\mathrm{QCoh}(X, S_{\mathcal{O}_X}(\mathcal{E}^\vee))$ the category of modules over the sheaf of algebras $S_{\mathcal{O}_X}(\mathcal{E}^\vee)$ which are quasi-coherent over $\mathcal{O}_X$, and by $\mathrm{Coh}(X, S_{\mathcal{O}_X}(\mathcal{E}^\vee))$ the full subcategory of the category $\mathrm{QCoh}(X, S_{\mathcal{O}_X}(\mathcal{E}^\vee))$ whose objects are locally finitely generated over $S_{\mathcal{O}_X}(\mathcal{E}^\vee)$. As p is an affine morphism, the direct image functor p_* induces equivalences of categories (see [Gro61a, 1.4.3]):

$$\begin{array}{rcl} \mathrm{QCoh}(E) & \xrightarrow{\sim} & \mathrm{QCoh}(X, S_{\mathcal{O}_X}(\mathcal{E}^\vee)), \\ \mathrm{Coh}(E) & \xrightarrow{\sim} & \mathrm{Coh}(X, S_{\mathcal{O}_X}(\mathcal{E}^\vee)). \end{array} \tag{2.3.5}$$

Using arguments similar to those of [Bor87, VI.2.11], $\mathcal{D}^b\mathrm{Coh}(X, S_{\mathcal{O}_X}(\mathcal{E}^\vee))$ identifies with the full subcategory of $\mathcal{D}^b\mathrm{QCoh}(X, S_{\mathcal{O}_X}(\mathcal{E}^\vee))$ whose objects have their cohomology sheaves in $\mathrm{Coh}(X, S_{\mathcal{O}_X}(\mathcal{E}^\vee))$. Now, a theorem of Bernstein (see [Bor87, VI.2.10]) ensures that $\mathcal{D}^b\mathrm{QCoh}(X, S_{\mathcal{O}_X}(\mathcal{E}^\vee))$ is equivalent to the full subcategory of $\mathcal{D}^b\mathrm{Mod}(X, S_{\mathcal{O}_X}(\mathcal{E}^\vee))$ (the bounded derived category of the category of *all* sheaves of $S_{\mathcal{O}_X}(\mathcal{E}^\vee)$-modules) whose objects have quasi-coherent cohomology. Combined with (2.3.5), this gives that $\mathcal{D}^b\mathrm{Coh}(E)$ is equivalent to the full subcategory of $\mathcal{D}^b\mathrm{Mod}(X, S_{\mathcal{O}_X}(\mathcal{E}^\vee))$ whose objects have their cohomology in $\mathrm{Coh}(X, S_{\mathcal{O}_X}(\mathcal{E}^\vee))$. Finally, using truncation functors, this latter subcategory identifies with the category $\mathcal{D}^{\mathrm{qc,fg}}(X, S_{\mathcal{O}_X}(\mathcal{E}^\vee))$, where $S_{\mathcal{O}_X}(\mathcal{E}^\vee)$ is considered as a dg-algebra concentrated in degree 0, with trivial differential. □

Recall that we have defined above, before (2.3.1), the bigraded dg-algebra $\mathcal{S}$. Let us also introduce the following $\mathbb{G}_\mathbf{m}$-dg-algebra with trivial differential:

$$\mathcal{R} := S_{\mathcal{O}_X}(\mathcal{F}^\vee) \qquad \text{with } \mathcal{F}^\vee \text{ in bidegree } (0,-2).$$

We have equivalences of categories ("regrading"):

$$\xi : \mathcal{C}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}) \xrightarrow{\sim} \mathcal{C}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{R}), \quad \xi : \mathcal{D}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}) \xrightarrow{\sim} \mathcal{D}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{R})$$

sending the $\mathcal{S}$-dg-module M to the $\mathcal{R}$-dg-module defined by $\xi(M)^i_j := M^{i-j}_j$ (with the same action of $S_{\mathcal{O}_X}(\mathcal{F}^\vee)$, and the same differential). The composition of the inclusion $\mathcal{D}^{+,\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}) \to \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S})$ and of ξ gives a functor $\mathcal{D}^{+,\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}) \to \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{R})$. Hence, using the analogue for F of equivalence 2.3.4, we obtain a functor

$$\mathcal{D}^{+,\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}) \to \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(F). \tag{2.3.6}$$

For this reason, we consider the category $\mathcal{D}^{+,\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S})$ as a "graded version" of the category $\mathcal{D}^b\mathrm{Coh}(F)$, and denote it

$$\mathrm{DGCoh}^{\mathrm{gr}}(F) := \mathcal{D}^{+,\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}). \tag{2.3.7}$$

Note that there exists a natural forgetful functor

$$\mathrm{For} : \mathrm{DGCoh}^{\mathrm{gr}}(F) \to \mathcal{D}^b\mathrm{Coh}(F), \tag{2.3.8}$$

the composition of (2.3.6) with the forgetful functor from $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(F)$ to $\mathcal{D}^b\mathrm{Coh}(F)$ or, equivalently, the composition

$$\mathcal{D}^{+,\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}) \to \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}) \cong \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X,\mathcal{R}) \to \mathcal{D}^{\mathrm{qc,fg}}(X,\mathcal{R}) \cong \mathcal{D}^b\mathrm{Coh}(F)$$

(the last equivalence is (2.3.3) applied to F).

Now, consider the dg-scheme $F^\perp \overset{R}{\cap}_{E^*} X$. As a module over $q_*\mathcal{O}_{E^*} \cong S_{\mathcal{O}_X}(\mathcal{E})$, $q_*\mathcal{O}_{F^\perp}$ is isomorphic to the quotient $S_{\mathcal{O}_X}(\mathcal{E})/(\mathcal{F}\cdot S_{\mathcal{O}_X}(\mathcal{E}))$. Hence it has a Koszul resolution

$$S_{\mathcal{O}_X}(\mathcal{E}) \otimes_{\mathcal{O}_X} \Lambda_{\mathcal{O}_X}(\mathcal{F}) \xrightarrow{\mathrm{qis}} S_{\mathcal{O}_X}(\mathcal{E})/(\mathcal{F}\cdot S_{\mathcal{O}_X}(\mathcal{E})),$$

where the generators of $\Lambda_{\mathcal{O}_X}(\mathcal{F})$ are in degree -1. Using Remark 1.8.6, we deduce an equivalence of categories

$$\mathrm{DGCoh}(F^\perp \overset{R}{\cap}_{E^*} X) \cong \mathcal{D}^{\mathrm{qc,fg}}(X,\mathcal{T}).$$

We are also interested in the "graded version"

$$\mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} X) := \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X,\mathcal{T}). \tag{2.3.9}$$

By definition we have a natural forgetful functor

$$\mathrm{For} : \mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} X) \to \mathrm{DGCoh}(F^\perp \overset{R}{\cap}_{E^*} X). \tag{2.3.10}$$

Finally, with notations (2.3.7) and (2.3.9), equivalence (2.3.1) gives the following result:

Theorem 2.3.11. *There exists an equivalence of categories, called* linear Koszul duality*,*

$$\mathrm{DGCoh}^{\mathrm{gr}}(F) \cong \mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} X).$$

Remark 2.3.12. Finally we have the following diagram:

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}(F) \arrow[rr, leftrightarrow, "\sim", "2.3.11"'] \arrow[d, "(2.3.8)"', "\mathrm{For}"] & & \mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} X) \arrow[d, "\mathrm{For}"', "(2.3.10)"] \\
\mathcal{D}^b\mathrm{Coh}(F) & & \mathrm{DGCoh}(F^\perp \overset{R}{\cap}_{E^*} X).
\end{tikzcd}$$

In sections 8 and 9 we will use this "correspondence", in the case $X = (G/B)^{(1)}$, $E = (\mathfrak{g}^* \times G/B)^{(1)}$, $F = \widetilde{\mathcal{N}}^{(1)}$ (see 3.1 for the notation), to relate certain simple restricted $\mathcal{U}\mathfrak{g}$-modules to certain indecomposable projective modules (see the discussion after Proposition 3.3.14 for details).

2.4 Linear Koszul duality and base change

Let X and Y be two non-singular varieties, and $\pi : X \to Y$ a *flat* and *proper* morphism between them. Let E be a vector bundle over Y, and $F \subset E$ a sub-bundle. Let $\mathcal{E}$ and $\mathcal{F}$ be their respective sheaves of sections. We will also consider the vector bundles $E_X := E \times_Y X$ and $F_X := F \times_Y X$ over X. Their sheaves of sections are respectively $\pi^*\mathcal{E}$ and $\pi^*\mathcal{F}$ (see [Gro61a, 1.7.11]). We denote by $\pi_F : F_X \to F$ the morphism induced by π. We consider the following $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebras with trivial differential:

$$\begin{array}{lll} \mathcal{S}_Y := S_{\mathcal{O}_Y}(\mathcal{F}^\vee), & \mathcal{S}_X := S_{\mathcal{O}_X}(\pi^*\mathcal{F}^\vee), & \text{with } \mathcal{F}^\vee \text{ in bidegree } (2,-2); \\ \mathcal{R}_Y := S_{\mathcal{O}_Y}(\mathcal{F}^\vee), & \mathcal{R}_X := S_{\mathcal{O}_X}(\pi^*\mathcal{F}^\vee), & \text{with } \mathcal{F}^\vee \text{ in bidegree } (0,-2); \\ \mathcal{T}_Y := \Lambda_{\mathcal{O}_Y}(\mathcal{F}), & \mathcal{T}_X := \Lambda_{\mathcal{O}_X}(\pi^*\mathcal{F}), & \text{with } \mathcal{F} \text{ in bidegree } (-1,2). \end{array}$$

In this situation we have two Koszul dualities (see 2.3.11):

$$\begin{array}{rcl} \kappa_Y : \quad \mathrm{DGCoh}^{\mathrm{gr}}(F) & \xrightarrow{\sim} & \mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} Y), \\ \kappa_X : \quad \mathrm{DGCoh}^{\mathrm{gr}}(F_X) & \xrightarrow{\sim} & \mathrm{DGCoh}^{\mathrm{gr}}(F_X^\perp \overset{R}{\cap}_{E_X^*} X). \end{array}$$

In this subsection we construct functors fitting in the following diagram:

$$\begin{tikzcd} \mathrm{DGCoh}^{\mathrm{gr}}(F_X) \arrow[r, shift left, "R(\tilde{\pi}_{\mathbb{G}_{\mathbf{m}}})_*"] \arrow[d, "\kappa_X", "\wr"'] & \mathrm{DGCoh}^{\mathrm{gr}}(F) \arrow[l, shift left, "L(\tilde{\pi}_{\mathbb{G}_{\mathbf{m}}})^*"] \arrow[d, "\kappa_Y", "\wr"'] \\ \mathrm{DGCoh}^{\mathrm{gr}}(F_X^\perp \overset{R}{\cap}_{E_X^*} X) \arrow[r, shift left, "R(\hat{\pi}_{\mathbb{G}_{\mathbf{m}}})_*"] & \mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} Y), \arrow[l, shift left, "L(\hat{\pi}_{\mathbb{G}_{\mathbf{m}}})^*"] \end{tikzcd}$$

and prove some compatibility results.

First, consider the categories on the right hand side of equivalences κ_Y, κ_X. Recall that, by definition,

$$\mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} Y) \cong \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_{\mathbf{m}}}(Y, \mathcal{T}_Y), \tag{2.4.1}$$

$$\mathrm{DGCoh}^{\mathrm{gr}}(F_X^\perp \overset{R}{\cap}_{E_X^*} X) \cong \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}_X). \tag{2.4.2}$$

The morphism π induces a morphism of $\mathbb{G}_{\mathbf{m}}$-equivariant dg-ringed spaces

$$\hat{\pi} : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y).$$

In subsection 1.7 we have constructed functors

$$\begin{array}{c} R(\hat{\pi}_{\mathbb{G}_{\mathbf{m}}})_* : \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}_X) \to \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(Y, \mathcal{T}_Y), \\ L(\hat{\pi}_{\mathbb{G}_{\mathbf{m}}})^* : \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(Y, \mathcal{T}_Y) \to \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}_X). \end{array}$$

As $\pi^*(\mathcal{T}_Y) \cong \mathcal{T}_X$, $\hat{\pi}^*$ identifies with π^*, and similarly for the $\mathbb{G}_{\mathbf{m}}$-analogues, *i.e.* the following diagram is commutative, where the vertical arrows are the forgetful functors:

$$\begin{tikzcd} \mathcal{C}_{\mathbb{G}_{\mathbf{m}}}(Y, \mathcal{T}_Y) \arrow[r, "(\hat{\pi}_{\mathbb{G}_{\mathbf{m}}})^*"] \arrow[d, "\mathrm{For}"'] & \mathcal{C}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}_X) \arrow[d, "\mathrm{For}"] \\ \mathcal{C}_{\mathbb{G}_{\mathbf{m}}}(Y, \mathcal{O}_Y) \arrow[r, "(\pi_{\mathbb{G}_{\mathbf{m}}})^*"] & \mathcal{C}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{O}_X). \end{tikzcd}$$

As $(\pi_{\mathbb{G}_\mathbf{m}})^*$ is exact (because π is flat), $(\hat{\pi}_{\mathbb{G}_\mathbf{m}})^*$ also is, and the corresponding diagram of derived categories and derived functors is also commutative. As $\Lambda_{\mathcal{O}_Y}(\mathcal{F})$ is a locally finitely generated module over $\mathcal{O}_Y$, a $\Lambda_{\mathcal{O}_Y}(\mathcal{F})$-module is locally finitely generated if and only if it is locally finitely generated over $\mathcal{O}_Y$. The same is true for $\Lambda_{\mathcal{O}_X}(\pi^*\mathcal{F})$. We deduce that $L(\hat{\pi}_{\mathbb{G}_\mathbf{m}})^*$ restricts to a functor from $\mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} Y)$ to $\mathrm{DGCoh}^{\mathrm{gr}}(F_X^\perp \overset{R}{\cap}_{E_X^*} X)$, via equivalences (2.4.1) and (2.4.2). Similarly, the functor $L(\hat{\pi})^*$ restricts to a functor $\mathrm{DGCoh}(F^\perp \overset{R}{\cap}_{E^*} Y) \to \mathrm{DGCoh}(F_X^\perp \overset{R}{\cap}_{E_X^*} X)$, and the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} Y) \arrow[r, "L(\hat{\pi}_{\mathbb{G}_\mathbf{m}})^*"] \arrow[d, "\mathrm{For}"'] & \mathrm{DGCoh}^{\mathrm{gr}}(F_X^\perp \overset{R}{\cap}_{E_X^*} X) \arrow[d, "\mathrm{For}"] \\
\mathrm{DGCoh}(F^\perp \overset{R}{\cap}_{E^*} Y) \arrow[r, "L(\hat{\pi})^*"] & \mathrm{DGCoh}(F_X^\perp \overset{R}{\cap}_{E_X^*} X).
\end{tikzcd}$$

We have seen in 1.7 that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{D}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{T}_X) \arrow[r, "R(\hat{\pi}_{\mathbb{G}_\mathbf{m}})_*"] \arrow[d, "\mathrm{For}"] & \mathcal{D}_{\mathbb{G}_\mathbf{m}}(Y, \mathcal{T}_Y) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}(X, \mathcal{T}_X) \arrow[r, "R(\hat{\pi})_*"] \arrow[d, "\mathrm{For}"] & \mathcal{D}(Y, \mathcal{T}_Y) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}(X, \mathcal{O}_X) \arrow[r, "R\pi_*"] & \mathcal{D}(Y, \mathcal{O}_Y).
\end{tikzcd}$$

As π is proper, we deduce as above that the functors $R(\hat{\pi})_*$ and $R(\hat{\pi}_{\mathbb{G}_\mathbf{m}})_*$ restrict to functors between the full subcategories whose objects have quasi-coherent, locally finitely generated cohomology (use [Har66, II.2.2]). Moreover, the following diagram commutes (using equivalences (2.4.1) and (2.4.2)):

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}(F_X^\perp \overset{R}{\cap}_{E_X^*} X) \arrow[r, "R(\hat{\pi}_{\mathbb{G}_\mathbf{m}})_*"] \arrow[d, "\mathrm{For}"'] & \mathrm{DGCoh}^{\mathrm{gr}}(F^\perp \overset{R}{\cap}_{E^*} Y) \arrow[d, "\mathrm{For}"] \\
\mathrm{DGCoh}(F_X^\perp \overset{R}{\cap}_{E_X^*} X) \arrow[r, "R(\hat{\pi})_*"] & \mathrm{DGCoh}(F^\perp \overset{R}{\cap}_{E^*} Y).
\end{tikzcd}$$

As a step towards the categories $\mathrm{DGCoh}^{\mathrm{gr}}(F)$ and $\mathrm{DGCoh}^{\mathrm{gr}}(F_X)$, we now study the categories $\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_X)$ and $\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(Y, \mathcal{S}_Y)$. The morphism π induces a morphism of $\mathbb{G}_\mathbf{m}$-equivariant dg-ringed spaces

$$\tilde{\pi} : (X, \mathcal{S}_X) \to (Y, \mathcal{S}_Y).$$

The $\mathbb{G}_\mathbf{m}$-equivariant dg-algebras $\mathcal{S}_X$ and $\mathcal{S}_Y$ are *not* non-positively graded. But we have seen in Remark 1.7.9 that the following derived functors are well defined:

$$R(\tilde{\pi}_{\mathbb{G}_\mathbf{m}})_* : \mathcal{D}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_X) \to \mathcal{D}_{\mathbb{G}_\mathbf{m}}(Y, \mathcal{S}_Y),$$
$$L(\tilde{\pi}_{\mathbb{G}_\mathbf{m}})^* : \mathcal{D}_{\mathbb{G}_\mathbf{m}}(Y, \mathcal{S}_Y) \to \mathcal{D}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_X).$$

As above, we will show that these functors restrict to functors between the full subcategories whose objects have quasi-coherent, locally finitely generated cohomology, and that the natural diagrams commute.

As $\pi^*\mathcal{S}_Y \cong \mathcal{S}_X$, the functor $(\tilde{\pi}_{\mathbb{G}_\mathbf{m}})^*$ is exact, and corresponds to $\pi^* : \mathcal{D}(Y, \mathcal{O}_Y) \to \mathcal{D}(X, \mathcal{O}_X)$ under the forgetful functor. Hence it restricts to a functor $\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(Y, \mathcal{S}_Y) \to \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_X)$. Moreover, the following diagram is clearly commutative (see (2.3.4) for the second vertical arrows):

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(Y, \mathcal{S}_Y) \arrow[r, "L(\tilde{\pi}_{\mathbb{G}_\mathbf{m}})^*"] \arrow[d, "\xi_Y"', "\wr"] & \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_X) \arrow[d, "\wr"', "\xi_X"] \\
\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(Y, \mathcal{R}_Y) \arrow[d, leftrightarrow, "\wr"'] & \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{R}_X) \arrow[d, leftrightarrow, "\wr"] \\
\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(F) \arrow[d, "\mathrm{For}"'] & \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(F_X) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}^b\mathrm{Coh}(F) \arrow[r, "L(\pi_F)^*"] & \mathcal{D}^b\mathrm{Coh}(F_X).
\end{tikzcd} \tag{2.4.3}$$

Now, consider the functor $R(\tilde{\pi}_{\mathbb{G}_\mathbf{m}})_*$. If $\mathcal{F}$ is in the category $\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_X)$, then $\xi_X(\mathcal{F})$ is in $\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{R}_X)$, and $\mathrm{For} \circ \xi_X(\mathcal{F})$ in $\mathcal{D}^{\mathrm{qc,fg}}(X, \mathcal{R}_X) \cong \mathcal{D}^b\mathrm{Coh}(F_X)$ (this equivalence is an analogue of (2.3.3)). Hence, as π_F is proper, $R(\pi_F)_* \circ \mathrm{For} \circ \xi_X(\mathcal{F})$ is in $\mathcal{D}^b\mathrm{Coh}(F)$. But this object coincides by construction with the object $\mathrm{For} \circ \xi_Y \circ R(\tilde{\pi}_{\mathbb{G}_\mathbf{m}})_*\mathcal{F}$ of $\mathcal{D}(Y, \mathcal{R}_Y)$. Hence $R(\tilde{\pi}_{\mathbb{G}_\mathbf{m}})_*\mathcal{F}$ belongs to the subcategory $\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(Y, \mathcal{S}_Y)$ of $\mathcal{D}_{\mathbb{G}_\mathbf{m}}(Y, \mathcal{S}_Y)$. This proves that $R(\tilde{\pi}_{\mathbb{G}_\mathbf{m}})_*$ restricts to a functor between $\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_X)$ and $\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(Y, \mathcal{S}_Y)$, and also that the analogue of diagram (2.4.3) for $R(\tilde{\pi}_{\mathbb{G}_\mathbf{m}})_*$ and $R(\pi_F)_*$ commutes.

Now we extend these results to the categories of *bounded below* $\mathbb{G}_\mathbf{m}$-dg-modules.

Lemma 2.4.4. *The functors*

$$(\tilde{\pi}^+_{\mathbb{G}_\mathbf{m}})_* : \mathcal{C}^+_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_X) \to \mathcal{C}^+_{\mathbb{G}_\mathbf{m}}(Y, \mathcal{S}_Y),$$
$$(\tilde{\pi}^+_{\mathbb{G}_\mathbf{m}})^* : \mathcal{C}^+_{\mathbb{G}_\mathbf{m}}(Y, \mathcal{S}_Y) \to \mathcal{C}^+_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_X)$$

admit a right, respectively left, derived functor. Moreover the following diagrams are commutative:

$$\begin{tikzcd}
\mathcal{D}^+_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_X) \arrow[r, "R(\tilde{\pi}^+_{\mathbb{G}_\mathbf{m}})_*"] \arrow[d] & \mathcal{D}^+_{\mathbb{G}_\mathbf{m}}(Y, \mathcal{S}_Y) \arrow[d] \\
\mathcal{D}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_X) \arrow[r, "R(\tilde{\pi}_{\mathbb{G}_\mathbf{m}})_*"] & \mathcal{D}_{\mathbb{G}_\mathbf{m}}(Y, \mathcal{S}_Y),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{D}^+_{\mathbb{G}_\mathbf{m}}(Y,\mathcal{S}_Y) \arrow[r, "L(\tilde{\pi}^+_{\mathbb{G}_\mathbf{m}})^*"] \arrow[d] & \mathcal{D}^+_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_X) \arrow[d] \\
\mathcal{D}_{\mathbb{G}_\mathbf{m}}(Y,\mathcal{S}_Y) \arrow[r, "L(\tilde{\pi}_{\mathbb{G}_\mathbf{m}})^*"] & \mathcal{D}_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_X).
\end{tikzcd}$$

Proof. The case of the inverse image functor is easy, and left to the reader (use the fact that π^* is exact). Consider the direct image functor $(\tilde{\pi}^+_{\mathbb{G}_\mathbf{m}})_*$. We have to show that this functor admits a right derived functor in the sense of Deligne ([Del73, 1.2]). But each object $\mathcal{M} \in \mathcal{C}^+_{\mathbb{G}_\mathbf{m}}(S_{\mathcal{O}_X}(\pi^*\mathcal{F}^\vee))$ admits a right resolution $\mathcal{I} \in \mathcal{C}^+_{\mathbb{G}_\mathbf{m}}(S_{\mathcal{O}_X}(\pi^*\mathcal{F}^\vee))$ all of whose components $\mathcal{I}^i_j$ are flabby (as sheaves on X). Indeed, consider for each i the Godement resolution (see [God64, II.4.3]) of the component $\bigoplus_j \mathcal{M}^i_j$. This defines a ($\mathbb{G}_\mathbf{m}$-equivariant) double complex with a ($\mathbb{G}_\mathbf{m}$-equivariant) action of $S_{\mathcal{O}_X}(\pi^*\mathcal{F}^\vee)$, all of whose components are flabby; taking the associated total complex gives the desired resolution. This dg-module $\mathcal{I}$ is $(\tilde{\pi}^+_{\mathbb{G}_\mathbf{m}})_*$-split on the right, hence the right derived functor is defined at $\mathcal{M}$.

By construction the following diagram is commutative, where For is the forgetful functor:

$$\begin{tikzcd}
\mathcal{D}^+_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_X) \arrow[r, "R(\tilde{\pi}^+_{\mathbb{G}_\mathbf{m}})_*"] \arrow[d, "\mathrm{For}"] & \mathcal{D}^+_{\mathbb{G}_\mathbf{m}}(Y,\mathcal{S}_Y) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}_{\mathbb{G}_\mathbf{m}}(X,\mathcal{O}_X) \arrow[r, "R\pi_*"] & \mathcal{D}_{\mathbb{G}_\mathbf{m}}(Y,\mathcal{O}_Y).
\end{tikzcd}$$

The commutativity of the diagram in the lemma follows (using Corollary 1.7.6). □

Using the results preceding this lemma, we deduce:

Corollary 2.4.5. *The functors $R(\tilde{\pi}^+_{\mathbb{G}_\mathbf{m}})_*$ and $L(\tilde{\pi}^+_{\mathbb{G}_\mathbf{m}})^*$ restrict to the subcategories whose objects have quasi-coherent, locally finitely generated cohomology. Moreover, recalling definitions* (2.3.7), (2.3.8), *the following diagrams commute:*

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}(F_X) \arrow[r, "R(\tilde{\pi}^+_{\mathbb{G}_\mathbf{m}})_*"] \arrow[d, "\mathrm{For}"'] & \mathrm{DGCoh}^{\mathrm{gr}}(F) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}^b\mathrm{Coh}(F_X) \arrow[r, "R(\pi_F)_*"] & \mathcal{D}^b\mathrm{Coh}(F),
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}(F) \arrow[r, "L(\tilde{\pi}^+_{\mathbb{G}_\mathbf{m}})^*"] \arrow[d, "\mathrm{For}"'] & \mathrm{DGCoh}^{\mathrm{gr}}(F_X) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}^b\mathrm{Coh}(F) \arrow[r, "L(\pi_F)^*"] & \mathcal{D}^b\mathrm{Coh}(F_X).
\end{tikzcd}$$

Because of these results, we will not write the superscript "+" on the functors associated to $\tilde{\pi}$ anymore. Now we study the compatibility of our functors.

Proposition 2.4.6. *Consider the following diagram:*

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}(F_X) \arrow[r, shift left, "R(\tilde{\pi}_{\mathbb{G}_{\mathbf{m}}})_*"] \arrow[d, "\kappa_X", "\wr"'] & \mathrm{DGCoh}^{\mathrm{gr}}(F) \arrow[l, shift left, "L(\tilde{\pi}_{\mathbb{G}_{\mathbf{m}}})^*"] \arrow[d, "\kappa_Y", "\wr"'] \\
\mathrm{DGCoh}^{\mathrm{gr}}(F_X^{\perp} \overset{R}{\cap}_{E_X^*} X) \arrow[r, shift left, "R(\hat{\pi}_{\mathbb{G}_{\mathbf{m}}})_*"] & \mathrm{DGCoh}^{\mathrm{gr}}(F^{\perp} \overset{R}{\cap}_{E^*} Y). \arrow[l, shift left, "L(\hat{\pi}_{\mathbb{G}_{\mathbf{m}}})^*"]
\end{tikzcd}$$

We have isomorphisms of functors

$$\begin{cases} R(\hat{\pi}_{\mathbb{G}_{\mathbf{m}}})_* \circ \kappa_X & \cong & \kappa_Y \circ R(\tilde{\pi}_{\mathbb{G}_{\mathbf{m}}})_*, \\ L(\hat{\pi}_{\mathbb{G}_{\mathbf{m}}})^* \circ \kappa_Y & \cong & \kappa_X \circ L(\tilde{\pi}_{\mathbb{G}_{\mathbf{m}}})^*. \end{cases}$$

Proof. The second isomorphism is easy, and left to the reader. The first one can be proved just like [Har66, II.5.6]. More precisely, let $\mathcal{M}$ be an object of $\mathrm{DGCoh}^{\mathrm{gr}}(F_X)$, with flabby components. Then $\kappa_Y \circ R(\tilde{\pi}_{\mathbb{G}_{\mathbf{m}}})_*(\mathcal{M}) \cong \mathcal{T}_Y^{\vee} \otimes_{\mathcal{O}_Y} \pi_* \mathcal{M}$. Next, by the projection formula (see *e.g.* [Har77, ex. II.5.1]), $(\mathcal{T}_Y)^{\vee} \otimes_{\mathcal{O}_Y} \pi_* \mathcal{M} \cong \pi_*(\mathcal{T}_X^{\vee} \otimes_{\mathcal{O}_X} \mathcal{M})$. Finally, as $R\pi_* = \mathrm{For} \circ R(\hat{\pi}_{\mathbb{G}_{\mathbf{m}}})_*$, one has a natural morphism $\pi_*(\mathcal{T}_X^{\vee} \otimes_{\mathcal{O}_X} \mathcal{M}) \to R(\hat{\pi}_{\mathbb{G}_{\mathbf{m}}})_*(\mathcal{T}_X^{\vee} \otimes_{\mathcal{O}_X} \mathcal{M})$. This defines a morphism of functors $\kappa_Y \circ R(\tilde{\pi}_{\mathbb{G}_{\mathbf{m}}})_* \to R(\hat{\pi}_{\mathbb{G}_{\mathbf{m}}})_* \circ \kappa_X$. To show that it is an isomorphism, as the question is local over Y, we can assume $\mathcal{F}$ is free. Then the result is clear. □

2.5 Linear Koszul duality and sub-bundles

Now we consider the following situation: $F_1 \subset F_2 \subset E$ are fiber bundles over the non-singular variety X. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the sheaves of sections of F_1, F_2. We define as above the $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebras with trivial differential:

$$\begin{array}{lll} \mathcal{S}_1 := S_{\mathcal{O}_X}(\mathcal{F}_1^{\vee}), & \mathcal{S}_2 := S_{\mathcal{O}_X}(\mathcal{F}_2^{\vee}), & \text{with } \mathcal{F}_i^{\vee} \text{ in bidegree } (2,-2), \\ \mathcal{R}_1 := S_{\mathcal{O}_X}(\mathcal{F}_1^{\vee}), & \mathcal{R}_2 := S_{\mathcal{O}_X}(\mathcal{F}_2^{\vee}), & \text{with } \mathcal{F}_i^{\vee} \text{ in bidegree } (0,-2), \\ \mathcal{T}_1 := \Lambda_{\mathcal{O}_X}(\mathcal{F}_1), & \mathcal{T}_2 := \Lambda_{\mathcal{O}_X}(\mathcal{F}_2), & \text{with } \mathcal{F}_i \text{ in bidegree } (-1,2). \end{array}$$

We have two Koszul dualities (see Theorem 2.3.11)

$$\begin{array}{llll} \kappa_1 : & \mathrm{DGCoh}^{\mathrm{gr}}(F_1) & \xrightarrow{\sim} & \mathrm{DGCoh}^{\mathrm{gr}}(F_1^{\perp} \overset{R}{\cap}_{E^*} X), \\ \kappa_2 : & \mathrm{DGCoh}^{\mathrm{gr}}(F_2) & \xrightarrow{\sim} & \mathrm{DGCoh}^{\mathrm{gr}}(F_2^{\perp} \overset{R}{\cap}_{E^*} X). \end{array}$$

The inclusion $f : F_1 \to F_2$ induces an injection $\mathcal{F}_1 \hookrightarrow \mathcal{F}_2$, and a surjection $\mathcal{F}_2^{\vee} \twoheadrightarrow \mathcal{F}_1^{\vee}$. Let

$$g : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$$

be the natural morphism of $\mathbb{G}_{\mathbf{m}}$-equivariant dg-ringed spaces. Our goal and strategy are the same as in 2.4.

Let us first consider the categories $\mathrm{DGCoh}^{\mathrm{gr}}(F_i^{\perp} \overset{R}{\cap}_{E^*} X)$. We have functors

$$\begin{array}{l} R(g_{\mathbb{G}_{\mathbf{m}}})_* : \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}_2) \to \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}_1), \\ L(g_{\mathbb{G}_{\mathbf{m}}})^* : \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}_1) \to \mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(X, \mathcal{T}_2). \end{array}$$

The functor $R(g_{\mathbb{G}_\mathbf{m}})_*$ is the "restriction of scalars" functor, and $L(g_{\mathbb{G}_\mathbf{m}})^*$ is the functor $\mathcal{M} \mapsto \Lambda_{\mathcal{O}_X}(\mathcal{F}_2) \otimes_{\Lambda_{\mathcal{O}_X}(\mathcal{F}_1)} \mathcal{M}$. Both are induced by exact functors on the abelian categories. It is clear that they preserve the conditions "qc, fg", and induce (see (2.3.9)) functors between the categories $\mathrm{DGCoh}^{\mathrm{gr}}(F_1^\perp \overset{R}{\cap}_{E^*} X)$ and $\mathrm{DGCoh}^{\mathrm{gr}}(F_2^\perp \overset{R}{\cap}_{E^*} X)$, and similarly for the non $\mathbb{G}_\mathbf{m}$-equivariant versions. Moreover, the following diagrams commute:

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}(F_2^\perp \overset{R}{\cap}_{E^*} X) \arrow[r, "R(g_{\mathbb{G}_\mathbf{m}})_*"] \arrow[d, "\mathrm{For}"'] & \mathrm{DGCoh}^{\mathrm{gr}}(F_1^\perp \overset{R}{\cap}_{E^*} X) \arrow[d, "\mathrm{For}"] \\
\mathrm{DGCoh}(F_2^\perp \overset{R}{\cap}_{E^*} X) \arrow[r, "Rg_*"] & \mathrm{DGCoh}(F_1^\perp \overset{R}{\cap}_{E^*} X),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}(F_1^\perp \overset{R}{\cap}_{E^*} X) \arrow[r, "L(g_{\mathbb{G}_\mathbf{m}})^*"] \arrow[d, "\mathrm{For}"'] & \mathrm{DGCoh}^{\mathrm{gr}}(F_2^\perp \overset{R}{\cap}_{E^*} X) \arrow[d, "\mathrm{For}"] \\
\mathrm{DGCoh}(F_1^\perp \overset{R}{\cap}_{E^*} X) \arrow[r, "Lg^*"] & \mathrm{DGCoh}(F_2^\perp \overset{R}{\cap}_{E^*} X).
\end{tikzcd}$$

Now, as a step towards the categories $\mathrm{DGCoh}^{\mathrm{gr}}(F_i)$, let us consider the categories $\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_i)$ $(i = 1, 2)$. We have a morphism of $\mathbb{G}_\mathbf{m}$-equivariant dg-ringed spaces

$$\tilde{f} : (X, \mathcal{S}_1) \to (X, \mathcal{S}_2)$$

and functors $R(\tilde{f}_{\mathbb{G}_\mathbf{m}})_*$ and $L(\tilde{f}_{\mathbb{G}_\mathbf{m}})^*$ (see again Remark 1.7.9). The functor $R(\tilde{f}_{\mathbb{G}_\mathbf{m}})_*$ is again the restriction of scalars. As $\mathcal{S}_2 \to \mathcal{S}_1$ is surjective, it restricts to the subcategories whose objects have quasi-coherent, locally finitely generated cohomology. Moreover, the following diagram, analogous to (2.4.3), is commutative:

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_1) \arrow[r, "R(\tilde{f}_{\mathbb{G}_\mathbf{m}})_*"] \arrow[d, "\xi_1"', "\wr"] & \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_2) \arrow[d, "\xi_2", "\wr"'] \\
\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{R}_1) \arrow[d, leftrightarrow, "\wr"'] & \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_2) \arrow[d, leftrightarrow, "\wr"] \\
\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(F_1) \arrow[d, "\mathrm{For}"'] & \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(F_2) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}^b\mathrm{Coh}(F_1) \arrow[r, "Rf_*"] & \mathcal{D}^b\mathrm{Coh}(F_2).
\end{tikzcd} \tag{2.5.1}$$

Consider the functor $L(\tilde{f}_{\mathbb{G}_\mathbf{m}})^*$. It is given by $\mathcal{M} \mapsto \mathcal{S}_1 \overset{L}{\otimes}_{\mathcal{S}_2} \mathcal{M}$. Arguments entirely similar to the ones used for the functor $R(\tilde{\pi}_{\mathbb{G}_\mathbf{m}})_*$ in 2.4 show that $L(\tilde{f}_{\mathbb{G}_\mathbf{m}})^*$ induces a functor from $\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_2)$ to $\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(X, \mathcal{S}_1)$, and that the diagram analogous to (2.5.1) commutes.

Let us extend these considerations to the categories of *bounded below* $\mathbb{G}_\mathbf{m}$-dg-modules.

Lemma 2.5.2. *The functors*

$$(\tilde{f}^+_{\mathbb{G}_\mathbf{m}})_* : \mathcal{C}^+_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_1) \to \mathcal{C}^+_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_2),$$
$$(\tilde{f}^+_{\mathbb{G}_\mathbf{m}})^* : \mathcal{C}^+_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_2) \to \mathcal{C}^+_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_1)$$

admit a right, respectively left, derived functor. Moreover, the following diagrams are commutative:

$$\begin{tikzcd}
\mathcal{D}^+_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_1) \arrow[r, "R(\tilde{f}^+_{\mathbb{G}_\mathbf{m}})_*"] \arrow[d] & \mathcal{D}^+_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_2) \arrow[d] \\
\mathcal{D}_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_1) \arrow[r, "R(\tilde{f}_{\mathbb{G}_\mathbf{m}})_*"] & \mathcal{D}_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_2),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{D}^+_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_2) \arrow[r, "L(\tilde{f}^+_{\mathbb{G}_\mathbf{m}})^*"] \arrow[d] & \mathcal{D}^+_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_1) \arrow[d] \\
\mathcal{D}_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_2) \arrow[r, "L(\tilde{f}_{\mathbb{G}_\mathbf{m}})^*"] & \mathcal{D}_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_1).
\end{tikzcd}$$

Proof. The case of the direct image functor is easy, and left to the reader. We define $\mathcal{F} := \mathcal{F}_1 \oplus \mathcal{F}_2$, and denote by $\mathcal{S}$ the $\mathbb{G}_\mathbf{m}$-equivariant dg-algebra $\mathcal{S} := S_{\mathcal{O}_X}(\mathcal{F}^\vee)$, with trivial differential and $\mathcal{F}^\vee$ in bidegree $(2,-2)$. Recall that $(\tilde{f}^+_{\mathbb{G}_\mathbf{m}})^*$ is the tensor product $\mathcal{M} \mapsto \mathcal{S}_1 \otimes_{\mathcal{S}_2} \mathcal{M}$. In this tensor product $\mathcal{S}_1$ is considered as a $\mathcal{S}_1$-$\mathcal{S}_2$-bimodule. As everything here is commutative, we can consider it as a module over $\mathcal{S}_1 \otimes_{\mathcal{O}_X} \mathcal{S}_2 \cong \mathcal{S}$. Now the natural morphism $\mathcal{S} \to \mathcal{S}_1$ is induced by the transpose of the diagonal embedding $\mathcal{F}_1 \hookrightarrow \mathcal{F}_1 \oplus \mathcal{F}_2$. Thus, if we denote by $\mathcal{G}$ the orthogonal of the image of $\mathcal{F}_1$ in this embedding, we have a (bounded below) Koszul resolution

$$\mathcal{S} \otimes_{\mathcal{O}_X} \Lambda_{\mathcal{O}_X}(\mathcal{G}) \xrightarrow{\mathrm{qis}} \mathcal{S}_1.$$

The first dg-module is K-flat over $\mathcal{S}$, which is itself K-flat over $\mathcal{S}_2$. Hence it is also K-flat over $\mathcal{S}_2$. Thus the tensor product with this dg-module defines a functor $L(\tilde{f}^+_{\mathbb{G}_\mathbf{m}})^* : \mathcal{D}^+_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_2) \to \mathcal{D}^+_{\mathbb{G}_\mathbf{m}}(X,\mathcal{S}_1)$. With this description, the commutativity of the corresponding diagram is obvious. $\square$

Exactly as for Corollary 2.4.5, we deduce:

Corollary 2.5.3. *The functors $R(\tilde{f}^+_{\mathbb{G}_\mathbf{m}})_*$ and $L(\tilde{f}^+_{\mathbb{G}_\mathbf{m}})^*$ restrict to the subcategories whose objects have quasi-coherent, locally finitely generated cohomology. Moreover, the following diagrams are commutative:*

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}(F_1) \arrow[r, "R(\tilde{f}^+_{\mathbb{G}_\mathbf{m}})_*"] \arrow[d, "\mathrm{For}"'] & \mathrm{DGCoh}^{\mathrm{gr}}(F_2) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}^b\mathrm{Coh}(F_1) \arrow[r, "Rf_*"] & \mathcal{D}^b\mathrm{Coh}(F_2)
\end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{DGCoh}^{\mathrm{gr}}(F_2) \arrow[r, "L(\tilde{f}^+_{\mathbb{G}_{\mathbf{m}}})^*"] \arrow[d, "\mathrm{For}"'] & \mathrm{DGCoh}^{\mathrm{gr}}(F_1) \arrow[d, "\mathrm{For}"] \\ \mathcal{D}^b\mathrm{Coh}(F_2) \arrow[r, "Lf^*"] & \mathcal{D}^b\mathrm{Coh}(F_1). \end{tikzcd}$$

As above, because of these results we will not write the superscript "+" on the functors associated to f anymore. Now we study the compatibility of these functors. Before that, let us make some remarks. From now on we assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are of constant rank, denoted by n_1 and n_2. We define $\mathcal{L}_i := \Lambda^{n_i}_{\mathcal{O}_X}(\mathcal{F}_i)$ for $i = 1, 2$. These are line bundles on X. One has isomorphisms $\psi_i : \mathcal{T}_i \to \mathcal{T}_i^\vee \otimes_{\mathcal{O}_X} \mathcal{L}_i[n_i]$, induced by the morphisms

$$\left\{ \begin{array}{ccc} \Lambda^j_{\mathcal{O}_X}(\mathcal{F}_i) \otimes_{\mathcal{O}_X} \Lambda^{n_i-j}_{\mathcal{O}_X}(\mathcal{F}_i) & \to & \mathcal{L}_i \\ t \otimes u & \mapsto & (-1)^{j(j+1)/2} t \wedge u. \end{array} \right.$$

Under this isomorphism the action of $\mathcal{T}_i$ on itself by left multiplication corresponds to the action on the dual defined as in 2.1, *i.e.* we have $\psi_i(st)(u) = (-1)^{\deg(s)(\deg(s)+1)/2}\psi_i(t)(su)$. We denote by $\langle 1 \rangle$ the shift in the $\mathbb{G}_{\mathbf{m}}$-grading defined by $(M\langle 1 \rangle)_n = M_{n-1}$, and by $\langle j \rangle$ its j-th power. This functor corresponds to the tensor product with the one-dimensional $\mathbb{G}_{\mathbf{m}}$-module corresponding to $\mathrm{Id}_{\mathbb{G}_{\mathbf{m}}}$. Taking the $\mathbb{G}_{\mathbf{m}}$-structure into account, ψ_i becomes an isomorphism $\mathcal{T}_i \cong \mathcal{T}_i^\vee \otimes_{\mathcal{O}_X} \mathcal{L}_i[n_i]\langle 2n_i \rangle$.

Proposition 2.5.4. *Consider the diagram*

$$\begin{tikzcd} \mathrm{DGCoh}^{\mathrm{gr}}(F_1) \arrow[r, shift left, "R(\tilde{f}_{\mathbb{G}_{\mathbf{m}}})_*"] \arrow[d, "\kappa_1", "\wr"'] & \mathrm{DGCoh}^{\mathrm{gr}}(F_2) \arrow[l, shift left, "L(\tilde{f}_{\mathbb{G}_{\mathbf{m}}})^*"] \arrow[d, "\kappa_2", "\wr"'] \\ \mathrm{DGCoh}^{\mathrm{gr}}(F_1^\perp \overset{R}{\cap}_{E^*} X) \arrow[r, shift left, "L(g_{\mathbb{G}_{\mathbf{m}}})^*"] & \mathrm{DGCoh}^{\mathrm{gr}}(F_2^\perp \overset{R}{\cap}_{E^*} X). \arrow[l, shift left, "R(g_{\mathbb{G}_{\mathbf{m}}})_*"] \end{tikzcd}$$

We have isomorphisms of functors

$$\left\{ \begin{array}{ccl} \kappa_1 \circ L(\tilde{f}_{\mathbb{G}_{\mathbf{m}}})^* & \cong & R(g_{\mathbb{G}_{\mathbf{m}}})_* \circ \kappa_2, \\ \kappa_2 \circ R(\tilde{f}_{\mathbb{G}_{\mathbf{m}}})_* & \cong & (L(g_{\mathbb{G}_{\mathbf{m}}})^* \circ \kappa_1) \otimes_{\mathcal{O}_X} \mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2^{-1}[n_1 - n_2]\langle 2n_1 - 2n_2 \rangle. \end{array} \right.$$

Proof. Let us begin with the first isomorphism. More precisely, we will construct an isomorphism of functors $L(\tilde{f}_{\mathbb{G}_{\mathbf{m}}})^* \circ (\kappa_2)^{-1} \cong (\kappa_1)^{-1} \circ R(g_{\mathbb{G}_{\mathbf{m}}})_*$. Recall the notation $\mathcal{F} := \mathcal{F}_1 \oplus \mathcal{F}_2$, $\mathcal{S} := S_{\mathcal{O}_X}(\mathcal{F}^\vee)$ and $\mathcal{G}$ introduced in the proof of Lemma 2.5.2. Let $\mathcal{N}$ be an object of $\mathrm{DGCoh}^{\mathrm{gr}}(F_2^\perp \overset{R}{\cap}_{E^*} X)$, which can be assumed to be bounded below (see Lemma 2.2.3). Then $(\kappa_1)^{-1} \circ R(g_{\mathbb{G}_{\mathbf{m}}})_*(\mathcal{N}) \cong \mathcal{S}_1 \otimes_{\mathcal{O}_X} \mathcal{N}$, where $\mathcal{N}$ is considered as a $\mathcal{T}_1$-dg-module. On the other hand,

$$\begin{array}{rcl} L(\tilde{f}_{\mathbb{G}_{\mathbf{m}}})^* \circ (\kappa_2)^{-1}(\mathcal{N}) & \cong & L(\tilde{f}_{\mathbb{G}_{\mathbf{m}}})^*(\mathcal{S}_2 \otimes_{\mathcal{O}_X} \mathcal{N}) \\ & \cong & (\mathcal{S} \otimes_{\mathcal{O}_X} \Lambda(\mathcal{G})) \otimes_{\mathcal{S}_2} (\mathcal{S}_2 \otimes_{\mathcal{O}_X} \mathcal{N}) \\ & \cong & (\mathcal{S} \otimes_{\mathcal{O}_X} \Lambda(\mathcal{G})) \otimes_{\mathcal{O}_X} \mathcal{N}. \end{array}$$

Hence there is a natural morphism of functors

$$L(\tilde{f}_{\mathbb{G}_\mathbf{m}})^* \circ (\kappa_2)^{-1} \to (\kappa_1)^{-1} \circ R(\tilde{g}_{\mathbb{G}_\mathbf{m}})_*,$$

induced by the morphism $\mathcal{S} \otimes_{\mathcal{O}_X} \Lambda(\mathcal{G}) \to \mathcal{S}_1$. We want to prove that it is an isomorphism. Using the exact sequence of dg-modules

$$0 \to \mathrm{Im}(d_{\mathcal{N}}) \to \mathcal{N} \to \mathcal{N}/\mathrm{Im}(d_{\mathcal{N}}) \to 0$$

we can assume, in addition to the fact that $\mathcal{N}$ is bounded below, that its differential is trivial (the dg-modules $\mathrm{Im}(d_{\mathcal{N}})$ and $\mathcal{N}/\mathrm{Im}(d_{\mathcal{N}})$ may not have quasi-coherent, locally finitely generated cohomology, but from now on in this proof we will not need any assumption on the cohomology of the dg-module).

Set $\mathcal{P} := \mathcal{S} \otimes_{\mathcal{O}_X} \Lambda(\mathcal{G})$. It is a K-flat $\mathcal{O}_X$-dg-module, as well as $\mathcal{S}_1$, and $\mathcal{P} \to \mathcal{S}_1$ is a quasi-isomorphism. We want to prove that the morphism $\mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{N} \to \mathcal{S}_1 \otimes_{\mathcal{O}_X} \mathcal{N}$ is a quasi-isomorphism, too. The differential on $\mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{N}$, respectively on $\mathcal{S}_1 \otimes_{\mathcal{O}_X} \mathcal{N}$, is the sum of the differential of $\mathcal{P}$, respectively of $\mathcal{S}_1$, and the Koszul differential d_{koszul} (recall that the differential of $\mathcal{S}_1$ is trivial). We consider $\mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{N}$ as the total complex of the double complex whose (p,q)-term is $\mathcal{P}^{q+2p} \otimes_{\mathcal{O}_X} \mathcal{N}^{-p}$, with first differential d_{koszul}, and second differential $d_{\mathcal{P}} \otimes \mathrm{Id}$. The first grading of this double complex is bounded above (because $\mathcal{N}$ is bounded below), hence the associated first spectral sequence converges (see [God64]). The same is true for $\mathcal{S}_1 \otimes_{\mathcal{O}_X} \mathcal{N}$ (in this case the second differential of the double complex is trivial). Hence we can forget the Koszul differential in these two complexes. Then the result follows from Lemma 1.3.6. This finishes the proof of the first isomorphism.

Let us now prove the second isomorphism. Let $\mathcal{M}$ be an object of $\mathrm{DGCoh}^{\mathrm{gr}}(F_1)$. We have $\kappa_2 \circ R(\tilde{f}_{\mathbb{G}_\mathbf{m}})_*(\mathcal{M}) \cong \mathcal{T}_2^\vee \otimes_{\mathcal{O}_X} \mathcal{M}$ (in the right hand side, $\mathcal{M}$ is considered as a $\mathcal{S}_2$-dg-module). Using the remarks before the statement of the proposition, one has an isomorphism of $\mathcal{T}_2$-dg-modules

$$\mathcal{T}_2^\vee \otimes_{\mathcal{O}_X} \mathcal{M} \cong (\mathcal{T}_2 \otimes_{\mathcal{O}_X} \mathcal{M}) \otimes_{\mathcal{O}_X} \mathcal{L}_2^{-1}[-n_2]\langle -2n_2 \rangle.$$

On the other hand, we have $L(g_{\mathbb{G}_\mathbf{m}})^* \circ \kappa_1(\mathcal{M}) \cong \mathcal{T}_2 \otimes_{\mathcal{T}_1} (\mathcal{T}_1^\vee \otimes_{\mathcal{O}_X} \mathcal{M})$, which, using the same remarks, is isomorphic to the dg-module $\mathcal{T}_2 \otimes_{\mathcal{O}_X} \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}_1^{-1}[-n_1]\langle -2n_1 \rangle$. This concludes the proof (one easily verifies that the differentials and the $\mathcal{T}_2$-module structures are compatible). □

3 Localization for restricted $\mathfrak{g}$-modules

In this section we prove localization theorems for restricted $\mathcal{U}\mathfrak{g}$-modules (see in particular Theorem 3.3.3).

3.1 Introduction

We use the same notation as in I.1.1. In particular, $\Bbbk$ is an algebraically closed field of characteristic p. In the rest of this chapter we assume that

$$p > h.$$

We denote by
$$C_0 := \{\lambda \in \mathbb{X} \mid \forall \alpha \in R^+,\ 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p\}$$
the set of integral weights in the fundamental alcove (which contains 0).

We will apply the results of section 2 on linear Koszul duality in the following situation. The base scheme X will be $\mathcal{B}^{(1)}$, the Frobenius twist of the flag variety of G (see *e.g.* [BMR08, 1.1.1] for Frobenius twists). The vector bundle will be $E = (\mathfrak{g}^* \times \mathcal{B})^{(1)}$, and the sub-bundle will be $\widetilde{\mathcal{N}}^{(1)} \subset (\mathfrak{g}^* \times \mathcal{B})^{(1)}$. Let $\mathcal{T}_{\mathcal{B}^{(1)}}$ denote the tangent bundle to $\mathcal{B}^{(1)}$. Its dual $\mathcal{T}^\vee_{\mathcal{B}^{(1)}}$ is the sheaf of sections of the vector bundle $\widetilde{\mathcal{N}}^{(1)}$ over $\mathcal{B}^{(1)}$.

Under our hypothesis $p > h$, there exists a G-equivariant isomorphism $\mathfrak{g}^* \cong \mathfrak{g}$, which induces an isomorphism $E \cong E^*$. Under this isomorphism, $(\widetilde{\mathcal{N}}^{(1)})^\perp$ identifies with $\widetilde{\mathfrak{g}}^{(1)}$. We thus obtain by Theorem 2.3.11 a Koszul duality

$$\kappa_{\mathcal{B}} : \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) \ \xrightarrow{\sim}\ \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}). \tag{3.1.1}$$

This equivalence is given by the following formula, for $\mathcal{M}$ in $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$:

$$\kappa_{\mathcal{B}}(\mathcal{M}) = (\Lambda(\mathcal{T}^\vee_{\mathcal{B}^{(1)}}))^\vee \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{M}.$$

We have an isomorphism $\Lambda^{\mathrm{top}}(\mathcal{T}^\vee_{\mathcal{B}^{(1)}}) \cong \mathcal{O}_{\mathcal{B}^{(1)}}(-2\rho)$. Hence, with the notation before Proposition 2.5.4 we have

$$(\Lambda(\mathcal{T}^\vee_{\mathcal{B}^{(1)}}))^\vee \cong \Lambda(\mathcal{T}^\vee_{\mathcal{B}^{(1)}}) \otimes \mathcal{O}_{\mathcal{B}^{(1)}}(2\rho)[-N]\langle -2N \rangle,$$

where $N = \mathrm{rk}(\mathcal{T}^\vee_{\mathcal{B}^{(1)}}) = \#R^+$. It follows that for $\mathcal{M}$ in $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$ we have

$$\kappa_{\mathcal{B}}(\mathcal{M}) = \Lambda(\mathcal{T}^\vee_{\mathcal{B}^{(1)}}) \otimes \mathcal{M} \otimes \mathcal{O}_{\mathcal{B}^{(1)}}(2\rho)[-N]\langle -2N \rangle. \tag{3.1.2}$$

In section 2 (see *e.g.* equation (2.3.9)) we have used the realization

$$\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \ \cong\ \mathcal{D}^{\mathrm{qc,fg}}(\mathcal{B}^{(1)},\, \Lambda_{\mathcal{O}_{\mathcal{B}^{(1)}}}(\mathcal{T}^\vee_{\mathcal{B}^{(1)}})) \tag{3.1.3}$$

where $\Lambda_{\mathcal{O}_{\mathcal{B}^{(1)}}}(\mathcal{T}^\vee_{\mathcal{B}^{(1)}})$ is considered as a dg-algebra with trivial differential, and $\mathcal{T}^\vee_{\mathcal{B}^{(1)}}$ in degree -1. Let $i : \widetilde{\mathfrak{g}}^{(1)} \hookrightarrow (\mathfrak{g}^* \times \mathcal{B})^{(1)}$ and $j : \mathcal{B}^{(1)} \hookrightarrow (\mathfrak{g}^* \times \mathcal{B})^{(1)}$ denote the closed embeddings. The realization (3.1.3) was constructed using a resolution of $i_*\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}$ over $\mathcal{O}_{(\mathfrak{g}^* \times \mathcal{B})^{(1)}}$. We can obtain another realization using a resolution of $j_*\mathcal{O}_{\mathcal{B}^{(1)}}$ over $\mathcal{O}_{(\mathfrak{g}^* \times \mathcal{B})^{(1)}}$, in particular the Koszul resolution

$$\mathcal{O}_{(\mathfrak{g}^* \times \mathcal{B})^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}) \ \xrightarrow{\mathrm{qis}}\ j_*\mathcal{O}_{\mathcal{B}^{(1)}}.$$

Using Remark 1.8.6 we deduce:

Proposition 3.1.4. *There exists an equivalence of triangulated categories*

$$\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \ \cong\ \mathcal{D}^{\mathrm{qc,fg}}(\widetilde{\mathfrak{g}}^{(1)},\, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$$

where $\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})$ is a dg-algebra with the generators of $\Lambda(\mathfrak{g}^{(1)})$ in degree -1, equipped with a Koszul differential.

From now on we will mainly use this realization of $\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$.

3.2 An equivalence of derived categories

In this subsection we prove an equivalence of derived categories that will be needed later. Recall the notations qc and fg introduced in subsection 2.2.

Let X be a variety, and let $\mathcal{Y}$ be a sheaf of dg-algebras on X which is non-positively graded and quasi-coherent as an $\mathcal{O}_X$-module. We also consider the sheaf of algebras $\mathcal{A} = \mathcal{Y}^0$. We have the coinduction functor, defined in 1.2:

$$\mathrm{Coind} : \left\{ \begin{array}{ccc} \mathcal{C}(X, \mathcal{O}_X) & \to & \mathcal{C}(X, \mathcal{Y}) \\ \mathcal{F} & \mapsto & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{Y}, \mathcal{F}). \end{array} \right.$$

Let Z be a closed subscheme of X. We denote by $\mathcal{D}^{\mathrm{qc}}_Z(X, \mathcal{Y})$ the full subcategory of $\mathcal{D}^{\mathrm{qc}}(X, \mathcal{Y})$ whose objects have their cohomology supported on Z (and similarly with qc replaced by qc, fg).

Lemma 3.2.1. *Let $\mathcal{F}$ be a $\mathcal{Y}$-dg-module which is quasi-coherent, supported on Z, and bounded below. There exists a K-injective $\mathcal{Y}$-dg-module $\mathcal{I}$, which is quasi-coherent and supported on Z, and a quasi-isomorphism $\mathcal{F} \xrightarrow{\mathrm{qis}} \mathcal{I}$.*

Proof. Let us first consider $\mathcal{F}$ as a complex of $\mathcal{O}_X$-modules. There exists a complex $\mathcal{J}_0$ of injective $\mathcal{O}_X$-modules, bounded below with the same bound as $\mathcal{F}$ and an injection of complexes of $\mathcal{O}_X$-modules $\mathcal{F} \hookrightarrow \mathcal{J}_0$ such that for any $n \in \mathbb{Z}$, $\mathcal{J}_0^n$ is quasi-coherent and supported on Z (see [Har66, II.7.18 and its proof]). By adjunction, this morphism induces an injection of $\mathcal{Y}$-dg-modules

$$\mathcal{F} \hookrightarrow \mathrm{Coind}(\mathcal{F}) \hookrightarrow \mathcal{I}_0 := \mathrm{Coind}(\mathcal{J}_0).$$

Moreover, $\mathcal{I}_0$ is still bounded below with the same bound as $\mathcal{F}$, and its components are quasi-coherent and supported on Z. The $\mathcal{O}_X$-dg-module $\mathcal{J}_0$ is K-injective (as a bounded below complex of injective $\mathcal{O}_X$-modules). Hence, by adjunction again, the $\mathcal{Y}$-dg-module $\mathcal{I}_0$ is K-injective.

Applying the same arguments to the cokernel of the morphism $\mathcal{F} \hookrightarrow \mathcal{I}_0$, and repeating, we obtain an exact sequence of $\mathcal{Y}$-dg-modules

$$\mathcal{F} \hookrightarrow \mathcal{I}_0 \to \mathcal{I}_1 \to \mathcal{I}_2 \to \cdots$$

where each $\mathcal{I}_j$ is K-injective, bounded below with a uniform bound, and its components are quasi-coherent and supported on Z. Now, as in the proof of Lemma 1.3.7, one proves that the natural morphism

$$\mathcal{F} \to \mathcal{I} := \mathrm{Tot}^{\oplus}(\cdots 0 \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots)$$

is a quasi-isomorphism, and that $\mathcal{I}$ is a K-injective $\mathcal{Y}$-dg-module. □

Lemma 3.2.2. *Let $\mathcal{F}$ be an object of $\mathcal{D}^{\mathrm{qc}}_Z(X, \mathcal{Y})$, whose cohomology is bounded. There exists a K-injective $\mathcal{Y}$-dg-module $\mathcal{G}$, which is quasi-coherent over $\mathcal{O}_X$ and supported on Z, and a quasi-isomorphism $\mathcal{F} \xrightarrow{\mathrm{qis}} \mathcal{G}$.*

Proof. Let us introduce a notation, to be used only in this proof. If $\mathcal{F}$ is a $\mathcal{Y}$-dg-module with bounded cohomology, we define

$$l(\mathcal{F}) := \max\{i \in \mathbb{Z} \mid H^i(\mathcal{F}) \neq 0\} - \min\{i \in \mathbb{Z} \mid H^i(\mathcal{F}) \neq 0\}$$

if $H(\mathcal{F}) \neq 0$, and $l(\mathcal{F}) = -1$ otherwise. We prove the lemma by induction on $l(\mathcal{F})$.

If $l(\mathcal{F}) = -1$, the result is obvious. Now let $n \geq 0$, and assume the result is true for any dg-module $\mathcal{G}$ with $l(\mathcal{G}) < n$. Let $\mathcal{F}$ be a $\mathcal{Y}$-dg-module with $l(\mathcal{F}) = n$. Let j be the lowest integer such that $H^j(\mathcal{F}) \neq 0$. Using a truncation functor, we can assume that $\mathcal{F}^k = 0$ for $k < j$. Then $\ker(d^j_{\mathcal{F}}) = H^j(\mathcal{F})$ is, by assumption, quasi-coherent and supported on Z. Let $\mathcal{K}$ denote the complex concentrated in degree j, with $\mathcal{K}^j = \ker(d^j_{\mathcal{F}})$. Then $\mathcal{K}$ is a sub-$\mathcal{Y}$-dg-module of $\mathcal{F}$. By Lemma 3.2.1, there exists a K-injective $\mathcal{Y}$-dg-module $\mathcal{I}_1$, quasi-coherent and supported on Z, and a quasi-isomorphism $i_1 : \mathcal{K} \xrightarrow{\text{qis}} \mathcal{I}_1$. Let $\mathcal{G}$ be the cokernel of the injection $\mathcal{K} \hookrightarrow \mathcal{F}$. Then $l(\mathcal{G}) < l(\mathcal{F})$. Hence, by induction, there exists a K-injective $\mathcal{Y}$-dg-module $\mathcal{I}_2$, quasi-coherent and supported on Z, and a quasi-isomorphism $i_2 : \mathcal{G} \xrightarrow{\text{qis}} \mathcal{I}_2$.

There exists a natural morphism $\mathcal{G}[-1] \to \mathcal{K}$ in $\mathcal{D}(X, \mathcal{Y})$, hence also a morphism $\mathcal{I}_2[-1] \to \mathcal{I}_1$ (since $\mathcal{I}_2$, resp. $\mathcal{I}_1$, is isomorphic to $\mathcal{G}$, resp. $\mathcal{K}$, in $\mathcal{D}(X, \mathcal{Y})$). By K-injectivity (see Definition 1.3.1), one can represent this morphism by an actual morphism of $\mathcal{Y}$-dg-modules $f : \mathcal{I}_2[-1] \to \mathcal{I}_1$ (unique up to homotopy). Let $\mathcal{I}_3$ be the cone of f. Then $\mathcal{I}_3$ is K-injective, quasi-coherent and supported on Z. We claim that there exists a quasi-isomorphism $\mathcal{F} \xrightarrow{\text{qis}} \mathcal{I}_3$. Indeed, in $\mathcal{D}(X, \mathcal{Y})$ we have the following diagram, where the lines are distinguished triangles:

$$\begin{tikzcd}
\mathcal{G}[-1] \arrow[r] \arrow[d, "i_2[-1]"] & \mathcal{K} \arrow[r] \arrow[d, "i_1"] & \mathcal{F} \arrow[d, dashed] \\
\mathcal{I}_2[-1] \arrow[r] & \mathcal{I}_1 \arrow[r] & \mathcal{I}_3.
\end{tikzcd}$$

The morphisms $i_2[-1]$ and i_1 can be completed to a morphism of triangles, yielding a morphism $i_3 : \mathcal{F} \to \mathcal{I}_3$ in $\mathcal{D}(X, \mathcal{Y})$. By K-injectivity of $\mathcal{I}_3$, i_3 can be realized as an actual morphism of $\mathcal{Y}$-dg-modules. Using the cohomology long exact sequence associated to a distinguished triangle and the five-lemma, i_3 is a quasi-isomorphism. This finishes the induction step, and the proof of the lemma. □

From now on we assume in addition:

$$\mathcal{Y} \text{ is coherent as an } \mathcal{O}_X\text{-module.}$$

In particular, as $\mathcal{A}$ is coherent over $\mathcal{O}_X$, an $\mathcal{A}$-module quasi-coherent over $\mathcal{O}_X$ is locally finitely generated over $\mathcal{A}$ if and only if it is coherent over $\mathcal{O}_X$. The same applies for $\mathcal{A}$ replaced by $H(\mathcal{Y})$, the cohomology of $\mathcal{Y}$.

Lemma 3.2.3. *Every $\mathcal{Y}$-dg-module $\mathcal{F}$ which is bounded, quasi-coherent over $\mathcal{O}_X$, and whose cohomology is coherent over $\mathcal{O}_X$ is the inductive limit of coherent sub-$\mathcal{Y}$-dg-modules which are quasi-isomorphic to $\mathcal{F}$ under the inclusion map.*

Proof. Our proof is similar to that of [Bor87, VI.2.11.(a)]. First, $\mathcal{F}$ is the inductive limit of coherent sub-dg-modules (this follows easily from the case of $\mathcal{O}_X$-modules), hence it is sufficient to show that given a coherent sub-dg-module $\mathcal{K}$ of $\mathcal{F}$, there exists a coherent sub-dg-module $\mathcal{G}$ of $\mathcal{F}$, containing $\mathcal{K}$ and quasi-isomorphic to $\mathcal{F}$ under the inclusion map.

This is proved by a simple (descending) induction. Let $j \in \mathbb{Z}$, and assume that we have found a subcomplex $\mathcal{G}_j$ of $\bigoplus_{i \geq j} \mathcal{F}^i$, coherent over $\mathcal{O}_X$, containing $\bigoplus_{i \geq j} \mathcal{K}^i$, such that $\mathcal{G}_j \to \mathcal{F}$ is a quasi-isomorphism in degrees greater than j and that $\mathcal{G}_j^j \cap \ker(d_{\mathcal{F}}^j) \to H^j(\mathcal{F})$ is surjective, and stable under $\mathcal{Y}$ (*i.e.* if $g \in \mathcal{G}_j^i$ and $y \in \mathcal{Y}^k$, and if $i + k \geq j$, then $y \cdot g \in \mathcal{G}_j^{i+k}$). Then we choose a sub-$\mathcal{A}$-module $\mathcal{N}^{j-1}$ of $\mathcal{F}^{j-1}$ containing $\mathcal{K}^{j-1}$, coherent over $\mathcal{O}_X$, whose image under $d_{\mathcal{F}}^{j-1}$ is $\mathcal{G}_j^j \cap \mathrm{Im}(d_{\mathcal{F}}^{j-1})$. Without altering these conditions, we can add a coherent sub-module of cocycles so that the new sub-module $\mathcal{N}^{j-1}$ contains representatives of all the elements of $H^{j-1}(\mathcal{F})$. We can also assume that $\mathcal{N}^{j-1}$ contains all the sections of the form $y \cdot g$ for $y \in \mathcal{Y}^i$ and $g \in \mathcal{G}_j^k$ with $i + k = j - 1$. Then we define $\mathcal{G}_{j-1}$ by

$$\mathcal{G}_{j-1}^k = \begin{cases} \mathcal{G}_j^k & \text{if } k \geq j, \\ \mathcal{N}^{j-1} & \text{if } k = j-1. \end{cases}$$

For j small enough, $\mathcal{G}_j$ is the desired sub-dg-module. $\square$

We denote by $\mathcal{C}_Z^{\mathrm{qc,fg}}(X, \mathcal{Y})$ the category of $\mathcal{Y}$-dg-modules which are coherent over $\mathcal{O}_X$ (this is equivalent to being quasi-coherent over $\mathcal{O}_X$ and locally finitely generated over $\mathcal{Y}$), and supported on Z. We denote by $\mathcal{D}\big(\mathcal{C}_Z^{\mathrm{qc,fg}}(X, \mathcal{Y})\big)$ the localization with respect to quasi-isomorphisms of the homotopy category of $\mathcal{C}_Z^{\mathrm{qc,fg}}(X, \mathcal{Y})$.

Proposition 3.2.4. *The functor*

$$\iota : \mathcal{D}\big(\mathcal{C}_Z^{\mathrm{qc,fg}}(X, \mathcal{Y})\big) \to \mathcal{D}_Z^{\mathrm{qc,fg}}(X, \mathcal{Y})$$

induced by the inclusion $\mathcal{C}_Z^{\mathrm{qc,fg}}(X, \mathcal{Y}) \hookrightarrow \mathcal{C}(X, \mathcal{Y})$ *is an equivalence of categories.*

Proof. This proof is again similar to the one in [Bor87, VI.2.11]. It follows easily from Lemmas 3.2.2 and 3.2.3, using truncation functors, that ι is essentially surjective.

Now, let us prove that it is full. Let $\mathcal{F}$ and $\mathcal{G}$ be objects of $\mathcal{C}_Z^{\mathrm{qc,fg}}(X, \mathcal{Y})$. In particular, $\mathcal{F}$ and $\mathcal{G}$ are bounded. A morphism $\phi : \iota(\mathcal{F}) \to \iota(\mathcal{G})$ in $\mathcal{D}_Z^{\mathrm{qc,fg}}(X, \mathcal{Y})$ is represented by a diagram

$$\iota(\mathcal{F}) \xrightarrow{\alpha} \mathcal{N} \xleftarrow{\beta} \iota(\mathcal{G})$$

where β is a quasi-isomorphism. Using Lemma 3.2.2 and truncation functors, one can assume that $\mathcal{N}$ is bounded, quasi-coherent, and supported on Z. By Lemma 3.2.3, there exists a coherent sub-dg-module $\mathcal{N}'$ of $\mathcal{N}$ (supported on Z), containing $\alpha(\mathcal{F})$ and $\beta(\mathcal{G})$, and quasi-isomorphic to $\mathcal{N}$ under the inclusion map. Then ϕ is represented by

$$\iota(\mathcal{F}) \xrightarrow{\alpha} \mathcal{N}' \xleftarrow{\beta} \iota(\mathcal{G}),$$

which is the image of a morphism in $\mathcal{D}\big(\mathcal{C}_Z^{\mathrm{qc,fg}}(X, \mathcal{Y})\big)$. Hence ι is full.

Finally we prove that ι is faithful. If a morphism $f : \mathcal{F} \to \mathcal{G}$ in $\mathcal{C}_Z^{\mathrm{qc,fg}}(X, \mathcal{Y})$ is such that $\iota(f) = 0$, then there exists $\mathcal{N}$ in $\mathcal{D}_Z^{\mathrm{qc,fg}}(X, \mathcal{Y})$, which can again be assumed to be bounded, quasi-coherent and supported on Z, and a quasi-isomorphism of $\mathcal{Y}$-dg-modules $g : \mathcal{G} \to \mathcal{N}$ such that $g \circ f$ is homotopic to zero. This homotopy is given by a morphism $h : \mathcal{F} \to \mathcal{N}[-1]$. By Lemma 3.2.3, there exists a coherent sub-dg-module $\mathcal{N}'$ of $\mathcal{N}$ containing $g(\mathcal{G})$ and $h(\mathcal{F})[1]$, and quasi-isomorphic to $\mathcal{N}$ under the inclusion. Replacing $\mathcal{N}$ by $\mathcal{N}'$, this proves that $f = 0$ in $\mathcal{D}\big(\mathcal{C}_Z^{\mathrm{qc,fg}}(X, \mathcal{Y})\big)$. The proof of the proposition is complete. □

3.3 Localization with a fixed Frobenius central character

Recall the notation and results of I.1.2. In [BMR08] and [BMR06] the authors give geometric counterparts for the derived categories of $\mathcal{U}\mathfrak{g}$-modules with a *generalized* Frobenius central character, and a fixed or generalized Harish-Chandra central character (see Theorem I.1.2.1). The relation between the Koszul duality (3.1.1) and representation theory is based on Theorem 3.3.3, which gives a geometric picture for the derived category of $\mathcal{U}\mathfrak{g}$-modules with a generalized (integral, regular) Harish-Chandra central character and a *fixed* trivial Frobenius central character.

Let us consider the derived intersection $(\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}$. As seen in Proposition 3.1.4, we have an equivalence of categories

$$\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \ \cong\ \mathcal{D}^{\mathrm{qc,fg}}(\widetilde{\mathfrak{g}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})). \tag{3.3.1}$$

Let $K_{\mathfrak{g}}$ denote the Koszul complex $S(\mathfrak{g}^{(1)}) \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})$, which is quasi-isomorphic to the trivial $S(\mathfrak{g}^{(1)})$-module $\Bbbk_0$. Here $S(\mathfrak{g}^{(1)})$ is in degree 0, and the generators of $\Lambda(\mathfrak{g}^{(1)})$ are in degree -1. By Poincaré-Birkhoff-Witt theorem, the enveloping algebra $\mathcal{U}\mathfrak{g}$ is free (hence flat) over $\mathfrak{Z}_{\mathrm{Fr}} \cong S(\mathfrak{g}^{(1)})$. Hence, if we consider $\mathcal{U}\mathfrak{g}$ as a sheaf of dg-algebras on $\mathrm{Spec}(\Bbbk)$, concentrated in degree 0, with trivial differential, there is a quasi-isomorphism of dg-algebras

$$\mathcal{U}\mathfrak{g} \otimes_{\mathfrak{Z}_{\mathrm{Fr}}} K_{\mathfrak{g}} \xrightarrow{\sim} \mathcal{U}\mathfrak{g} \otimes_{\mathfrak{Z}_{\mathrm{Fr}}} \Bbbk_0,$$

and hence an equivalence of categories (see Proposition 1.5.6):

$$\mathcal{D}\mathrm{Mod}((\mathcal{U}\mathfrak{g})_0) \ \cong\ \mathcal{D}(\mathrm{Spec}(\Bbbk), \mathcal{U}\mathfrak{g} \otimes_{\mathfrak{Z}_{\mathrm{Fr}}} K_{\mathfrak{g}}).$$

Restricting to the subcategories of objects with finitely generated cohomology, we obtain an equivalence:

$$\mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0) \ \cong\ \mathcal{D}^{\mathrm{fg}}(\mathrm{Spec}(\Bbbk), \mathcal{U}\mathfrak{g} \otimes_{\mathfrak{Z}_{\mathrm{Fr}}} K_{\mathfrak{g}}). \tag{3.3.2}$$

Here we have used that, as $(\mathcal{U}\mathfrak{g})_0$ is noetherian, the functor

$$\mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0) \ \to\ \mathcal{D}^{\mathrm{fg}}(\mathrm{Spec}(\Bbbk), (\mathcal{U}\mathfrak{g})_0)$$

is an equivalence. In the rest of this subsection, we write $\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})$ for the dg-algebra $\mathcal{U}\mathfrak{g} \otimes_{\mathfrak{Z}_{\mathrm{Fr}}} K_{\mathfrak{g}}$.

Then we have the following result, which completes Theorem I.1.2.1(i) for $\chi = 0$:

Theorem 3.3.3. *Let $\lambda \in \mathbb{X}$ be regular. There exists an equivalence of triangulated categories*

$$\widehat{\gamma}_\lambda^{\mathcal{B}} : \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \xrightarrow{\sim} \mathcal{D}^b\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0).$$

The proof of the theorem will occupy the rest of this subsection. We begin with several lemmas.

First, we have seen in the remarks following Theorem I.1.2.1 that the projection $\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^* \to \widetilde{\mathfrak{g}}^{(1)}$ induces an isomorphism between the formal neighborhood of $\mathcal{B}^{(1)} \times \{\lambda\}$ and the formal neighborhood of $\mathcal{B}^{(1)}$. We denote these formal neighborhoods by $\widehat{\mathcal{B}^{(1)}}$. To simplify notations, in this subsection we denote the variety $\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$ by X. Then we have:

Lemma 3.3.4. *The natural functor*

$$\mathcal{D}^{\mathrm{qc,fg}}(\widetilde{\mathfrak{g}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \to \mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)} \times \{\lambda\}}(X, \mathcal{O}_X \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$$

is an equivalence of categories.

Proof. First, we observe that any object of $\mathcal{D}^{\mathrm{qc,fg}}(\widetilde{\mathfrak{g}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$ has its cohomology supported on $\mathcal{B}^{(1)}$ (because $H^0(\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) = \mathcal{O}_{\mathcal{B}^{(1)}}$). Hence, by Proposition 3.2.4, the category $\mathcal{D}^{\mathrm{qc,fg}}(\widetilde{\mathfrak{g}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$ is equivalent to $\mathcal{D}\big(\mathcal{C}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big)$.

Now, as the formal neighborhoods of $\mathcal{B}^{(1)}$ in $\widetilde{\mathfrak{g}}^{(1)}$ and of $\mathcal{B}^{(1)} \times \{\lambda\}$ in X are isomorphic, the category $\mathcal{C}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$ is equivalent to $\mathcal{C}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)} \times \{\lambda\}}(X, \mathcal{O}_X \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$.

Finally, using Proposition 3.2.4 again, we obtain the result. □

We can consider $\mathcal{U}\mathfrak{g}$ as a sheaf of algebras either on the point $\mathrm{Spec}(\Bbbk)$, or on $\mathrm{Spec}(\mathfrak{Z}) \cong \mathfrak{g}^{*(1)} \times_{\mathfrak{h}^{*(1)}/W} \mathfrak{h}^*/(W, \bullet)$. It follows easily from Proposition 3.2.4 that the category

$$\mathcal{D}^{\mathrm{qc,fg}}(\mathfrak{g}^{*(1)} \times_{\mathfrak{h}^{*(1)}/W} \mathfrak{h}^*/(W, \bullet), \mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$$

is equivalent to $\mathcal{D}^{\mathrm{fg}}(\mathrm{Spec}(\Bbbk), \mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$. We denote this category simply by $\mathcal{D}^{\mathrm{fg}}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$. We also denote by $\mathcal{D}^{\mathrm{fg}}_\lambda(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$ the full subcategory of $\mathcal{D}^{\mathrm{fg}}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$ whose objects are the dg-modules M such that $\mathcal{U}\mathfrak{g}$ acts on $H(M)$ with generalized character $(\lambda, 0)$. It also follows from Proposition 3.2.4 that this category is equivalent to the localization of the homotopy category of finitely generated $\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})$-dg-modules on which $\mathcal{U}\mathfrak{g}$ acts with generalized character $(\lambda, 0)$. We also use the same notation and results for $\mathcal{U}\mathfrak{g}$ instead of $\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})$.

The following result follows easily from these definitions, using [BMR08, 1.3.7].

Lemma 3.3.5. *Equivalence* (3.3.2) *restricts to an equivalence of categories*

$$\mathcal{D}^b\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0) \cong \mathcal{D}^{\mathrm{fg}}_\lambda(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})).$$

Next, let us recall some results concerning dg-algebras. Let A be a dg-algebra (*i.e.* a sheaf of dg-algebras on $\mathrm{Spec}(\Bbbk)$). We use the same notation as in section 1, except that we omit "$\mathrm{Spec}(\Bbbk)$" in the notation for categories. An A-dg-module M is said to be *K-projective* if for any acyclic A-dg-module N, the complex of vector spaces $\mathrm{Hom}_A(M,N)$ is acyclic. By the results of [BL94, section 10], every A-dg-module has a left K-projective resolution. As in subsection 1.4, we deduce:

Lemma 3.3.6. *Any triangulated functor from $\mathcal{C}(A)$ to a triangulated category has a left derived functor in the sense of Deligne, which can be computed by means of K-projective resolutions.*

Proof of Theorem 3.3.3. We will show that the equivalences constructed in [BMR08] are "compatible with the tensor product with $K_{\mathfrak{g}}$".

First step: Let us prove the following equivalence of categories:

$$\mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}\times\{\lambda\}}(X,\, \mathcal{O}_X \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \ \cong\ \mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}\times\{\lambda\}}(X,\, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})). \tag{3.3.7}$$

It will follow from the results of [BMR08] coupled with Proposition 3.2.4, which allows us to consider nice abelian categories rather than derived categories with conditions on the cohomology.

As in [BMR08] we define the functors

$$F : \left\{ \begin{array}{ccc} \mathcal{C}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}\times\{\lambda\}}(X,\, \mathcal{O}_X \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) & \to & \mathcal{C}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}\times\{\lambda\}}(X,\, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \\ \mathcal{F} & \mapsto & \mathcal{M}^{\lambda} \otimes_{\mathcal{O}_{\widehat{\mathcal{B}^{(1)}}}} \mathcal{F} \end{array} \right. ,$$

$$G : \left\{ \begin{array}{ccc} \mathcal{C}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}\times\{\lambda\}}(X,\, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) & \to & \mathcal{C}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}\times\{\lambda\}}(X,\, \mathcal{O}_X \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \\ \mathcal{G} & \mapsto & \mathcal{H}om_{\widetilde{\mathcal{D}}}(\mathcal{M}^{\lambda}, \mathcal{G}) \end{array} \right. .$$

These functors are exact. There are natural morphisms of functors $F \circ G \to \mathrm{Id}$ and $\mathrm{Id} \to G \circ F$. These functors and morphisms of functors coincide with the ones considered in [BMR08, 5.1.1] under the forgetful functors

$$\begin{array}{c} \mathcal{C}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}\times\{\lambda\}}(X,\, \mathcal{O}_X \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \ \to\ \mathcal{C}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}\times\{\lambda\}}(X,\, \mathcal{O}_X) \cong \mathcal{C}^b\mathrm{Coh}_{\mathcal{B}^{(1)}\times\{\lambda\}}(X) \\ \mathcal{C}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}\times\{\lambda\}}(X,\, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \ \to\ \mathcal{C}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}\times\{\lambda\}}(X,\, \widetilde{\mathcal{D}}) \cong \mathcal{C}^b\mathrm{Mod}^{\mathrm{c}}_{(\lambda,0)}(\widetilde{\mathcal{D}}). \end{array}$$

Hence, by [BMR08, 5.1.1], the morphisms of functors $F \circ G \to \mathrm{Id}$ and $\mathrm{Id} \to G \circ F$ are isomorphisms, and F and G are equivalences of categories. They induce equivalences of the derived categories (3.3.7) (here we use Proposition 3.2.4).

Thus, combining (3.3.1), Lemma 3.3.4 and (3.3.7), we have obtained:

$$\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \ \cong\ \mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}\times\{\lambda\}}(X,\, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})). \tag{3.3.8}$$

Second step: Now we construct an equivalence of categories

$$\mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}\times\{\lambda\}}(X,\, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \ \cong\ \mathcal{D}^{\mathrm{fg}}_{\lambda}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})). \tag{3.3.9}$$

By the projection formula ([Har77, II.Ex.5.1]), we have

$$\Gamma(\widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \cong \Gamma(\widetilde{\mathcal{D}}) \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}) \cong \widetilde{U} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})$$

where $\widetilde{U} := \mathcal{U}\mathfrak{g} \otimes_{\mathfrak{Z}_{\mathrm{HC}}} S(\mathfrak{h})$ (see [BMR08, 3.4.1] for the second isomorphism). The dg-algebra $\widetilde{U} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})$ contains $\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})$ as a sub-dg-algebra. Hence (see 1.5) there exists a functor

$$R\Gamma : \mathcal{D}(X, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \ \to \ \mathcal{D}(\mathrm{Spec}(\Bbbk), \mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})).$$

Moreover, the following diagram is commutative (see Corollary 1.5.3):

$$\begin{tikzcd} \mathcal{D}(X, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \arrow[r, "R\Gamma"] \arrow[d, "\mathrm{For}"'] & \mathcal{D}(\mathrm{Spec}(\Bbbk), \mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \arrow[d, "\mathrm{For}"] \\ \mathcal{D}(X, \widetilde{\mathcal{D}}) \arrow[r, "R\Gamma"] & \mathcal{D}(\mathrm{Spec}(\Bbbk), \mathcal{U}\mathfrak{g}). \end{tikzcd} \tag{3.3.10}$$

Recall the notation introduced before Lemma 3.3.5. By Proposition 3.2.4 again, the functor $\mathcal{D}^b\mathrm{Mod}^{\mathrm{c}}_{(\lambda,0)}(\widetilde{\mathcal{D}}) \to \mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)} \times \{\lambda\}}(X, \widetilde{\mathcal{D}})$ is an equivalence of categories. If $\mathcal{F}$ is an object of the subcategory $\mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)} \times \{\lambda\}}(X, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$, then $\mathrm{For}(\mathcal{F})$ is in $\mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)} \times \{\lambda\}}(X, \widetilde{\mathcal{D}}) \cong \mathcal{D}^b\mathrm{Mod}^{\mathrm{c}}_{(\lambda,0)}(\widetilde{\mathcal{D}})$. Hence, by [BMR08, 3.1.9], $R\Gamma(\mathrm{For}(\mathcal{F}))$ is in the subcategory $\mathcal{D}^{\mathrm{fg}}_{\lambda}(\mathcal{U}\mathfrak{g})$. Using diagram (3.3.10), we deduce that $R\Gamma(\mathcal{F})$ is in $\mathcal{D}^{\mathrm{fg}}_{\lambda}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$. Hence we have proved that $R\Gamma$ induces a functor

$$R\Gamma : \mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)} \times \{\lambda\}}(X, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \to \mathcal{D}^{\mathrm{fg}}_{\lambda}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})).$$

Moreover, the following diagram is commutative:

$$\begin{tikzcd} \mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)} \times \{\lambda\}}(X, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \arrow[r, "R\Gamma"] \arrow[d, "\mathrm{For}"'] & \mathcal{D}^{\mathrm{fg}}_{\lambda}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \arrow[d, "\mathrm{For}"] \\ \mathcal{D}^b\mathrm{Mod}^{c}_{(\lambda,0)}(\widetilde{\mathcal{D}}) \arrow[r, "R\Gamma"] & \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,0)}(\mathcal{U}\mathfrak{g}). \end{tikzcd} \tag{3.3.11}$$

Now we construct an adjoint for this functor. First, consider

$$\mathrm{Loc}_K : \left\{ \begin{array}{ccc} \mathcal{C}(\mathrm{Spec}(\Bbbk), \mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) & \to & \mathcal{C}(X, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \\ M & \mapsto & \widetilde{\mathcal{D}} \otimes_{\mathcal{U}\mathfrak{g}} M \end{array} \right. .$$

Using Lemma 3.3.6, this functor admits a left derived functor

$$\mathcal{L}_K : \mathcal{D}(\mathrm{Spec}(\Bbbk), \mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \ \to \ \mathcal{D}(X, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$$

(which can be computed by means of K-projective resolutions). The following diagram is commutative:

$$\begin{tikzcd} \mathcal{D}(\mathrm{Spec}(\Bbbk), \mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \arrow[r, "\mathcal{L}_K"] \arrow[d, "\mathrm{For}"'] & \mathcal{D}(X, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \arrow[d, "\mathrm{For}"] \\ \mathcal{D}(\mathrm{Spec}(\Bbbk), \mathcal{U}\mathfrak{g}) \arrow[r, "\widetilde{\mathcal{D}} \overset{L}{\otimes}_{\mathcal{U}\mathfrak{g}} -"] & \mathcal{D}(X, \widetilde{\mathcal{D}}) \end{tikzcd} \tag{3.3.12}$$

where the bottom arrow is the usual derived tensor product. Indeed, both derived functors can be computed using K-projective resolutions, and every K-projective $\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})$-dg-module restricts to a K-projective complex of $\mathcal{U}\mathfrak{g}$-modules. (This follows from Lemma 1.2.3 and the fact that coinduction from $\mathcal{U}\mathfrak{g}$ to $\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})$ sends acyclic dg-modules to acyclic dg-modules, as $\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})$ is K-projective over $\mathcal{U}\mathfrak{g}$.)

As $\mathcal{U}\mathfrak{g}$ is noetherian, the natural morphism $\mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}(\mathcal{U}\mathfrak{g}) \to \mathcal{D}^{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$ is an equivalence of categories. Using this and diagram (3.3.12) we deduce (as above) that $\mathcal{L}_K$ induces a functor

$$\mathcal{L}_K : \mathcal{D}^{\mathrm{fg}}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \ \to \ \mathcal{D}^{\mathrm{qc,fg}}(X, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})).$$

Moreover, for any object M of $\mathcal{D}^{\mathrm{fg}}_{\lambda}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$ there is a canonical decomposition $\mathcal{L}_K(M) \cong \bigoplus_{\mu \in W \bullet \lambda} \mathcal{L}_K^{\lambda \to \mu}(M)$ with $\mathcal{L}_K^{\lambda \to \mu}(M)$ in $\mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)} \times \{\mu\}}(X, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$. Indeed, using Proposition 3.2.4, we have such a direct sum decomposition as a complex of $\widetilde{\mathcal{D}}$-modules (as in [BMR08, 3.3.1]). As the actions of $\Lambda(\mathfrak{g}^{(1)})$ and $\mathrm{S}(\mathfrak{h}) \subset \widetilde{\mathcal{D}}$ commute, each summand is in fact a sub-$\widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})$-dg-module.

Now we define $\mathcal{L}_K^{\widehat{\lambda}} := \mathcal{L}_K^{\lambda \to \lambda}$. Then by construction we have a commutative diagram

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{fg}}_{\lambda}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \arrow[r, "\mathcal{L}_K^{\widehat{\lambda}}"] \arrow[d, "\mathrm{For}"'] & \mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)} \times \{\lambda\}}(X, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,0)}(\mathcal{U}\mathfrak{g}) \arrow[r, "\mathcal{L}^{\widehat{\lambda}}"] & \mathcal{D}^b\mathrm{Mod}^{\mathrm{c}}_{(\lambda,0)}(\widetilde{\mathcal{D}})
\end{tikzcd} \tag{3.3.13}$$

where $\mathcal{L}^{\widehat{\lambda}}$ is the functor defined in [BMR08, 3.3.1].

As in [BMR08, 3.3.2] one proves that the functors $\mathcal{L}_K^{\widehat{\lambda}}$ and $R\Gamma$ form an adjoint pair. Hence there are adjunction morphisms $\mathrm{Id} \to R\Gamma \circ \mathcal{L}_K^{\widehat{\lambda}}$ and $\mathcal{L}_K^{\widehat{\lambda}} \circ R\Gamma \to \mathrm{Id}$, which coincide, under the natural forgetful functors, with the adjunction morphisms $\mathrm{Id} \to R\Gamma \circ \mathcal{L}^{\widehat{\lambda}}$ and $\mathcal{L}^{\widehat{\lambda}} \circ R\Gamma \to \mathrm{Id}$ of [BMR08]. In [BMR08, 3.6] the authors prove that the latter morphisms are isomorphisms. Hence the former morphisms also are isomorphisms. This concludes the proof of (3.3.9).

Recalling Lemma 3.3.5, we have proved equivalences:

$$\begin{aligned}
\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \ &\overset{(3.3.8)}{\cong} \ \mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)} \times \{\lambda\}}(X, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \\
&\overset{(3.3.9)}{\cong} \ \mathcal{D}^{\mathrm{fg}}_{\lambda}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \\
&\overset{3.3.5}{\cong} \ \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{\lambda}((\mathcal{U}\mathfrak{g})_0).
\end{aligned}$$

This concludes the proof of Theorem 3.3.3. □

Let $p : (\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)} \to \widetilde{\mathfrak{g}}^{(1)}$ be the natural morphism of dg-schemes. It can be realized as the natural morphism of dg-ringed spaces

$$(\widetilde{\mathfrak{g}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \ \to \ (\widetilde{\mathfrak{g}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}).$$

The following proposition is clear from our constructions (see in particular diagrams (3.3.11) and (3.3.13)):

Proposition 3.3.14. *The following diagram is commutative, where the functor* Incl *is induced by the inclusion* $\mathrm{Mod}^{\mathrm{fg}}_{\lambda}((\mathcal{U}\mathfrak{g})_0) \hookrightarrow \mathrm{Mod}^{\mathrm{fg}}_{(\lambda,0)}(\mathcal{U}\mathfrak{g})$*:*

$$\begin{tikzcd} \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[r, "Rp_*"] \arrow[d, "\widehat{\gamma}^{\mathcal{B}}_{\lambda}", "\wr"'] & \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}) \arrow[d, "\gamma^{\mathcal{B}}_{\lambda}", "\wr"'] \\ \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{\lambda}((\mathcal{U}\mathfrak{g})_0) \arrow[r, "\mathrm{Incl}"] & \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\lambda,0)}(\mathcal{U}\mathfrak{g}). \end{tikzcd}$$

Recall the Koszul duality $\kappa_{\mathcal{B}}$ of (3.1.1). The situation is the following, where $\lambda \in \mathbb{X}$ is regular:

$$(*) \qquad \begin{tikzcd} & \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, "\sim", "\kappa_{\mathcal{B}}"'] \arrow[d, "\mathrm{For}"', "(2.3.8)"] & \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[d, "(2.3.10)"', "\mathrm{For}"] \\ \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, hook] \arrow[d, leftrightarrow, "\wr"', "\text{Ch. I, (1.2.3)}"] & \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)}) & \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[d, leftrightarrow, "3.3.3"', "\wr"] \\ \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_0((\mathcal{U}\mathfrak{g})^{\lambda}) & & \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{\lambda}((\mathcal{U}\mathfrak{g})_0) \end{tikzcd}$$

Hence we have constructed some "correspondence" between $\mathcal{U}\mathfrak{g}$-modules with fixed trivial Frobenius character and generalized Harish-Chandra character λ (on the right hand side), and $\mathcal{U}\mathfrak{g}$-modules with generalized trivial Frobenius character and fixed Harish-Chandra character λ (on the left hand side). One of the main results of this chapter is that, under the assumption that p is large enough so that Lusztig's conjecture from [Lus80b] is true (see 0.5), "indecomposable projective modules correspond to simple modules" under this correspondence (see Theorem 4.4.3 below for a precise statement).

To finish this subsection, let us remark that entirely similar arguments give the following more general theorem:

Theorem 3.3.15. *Let* $\mu, \mathcal{P}$ *be as in* (ii) *of Theorem* I.1.2.1*. There exists an equivalence of triangulated categories*

$$\widehat{\gamma}^{\mathcal{P}}_{\mu} : \mathrm{DGCoh}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) \xrightarrow{\sim} \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{\mu}((\mathcal{U}\mathfrak{g})_0)$$

making the following diagram commutative, where Incl *is induced by the inclusion of categories* $\mathrm{Mod}^{\mathrm{fg}}_{\mu}((\mathcal{U}\mathfrak{g})_0) \hookrightarrow \mathrm{Mod}^{\mathrm{fg}}_{(\mu,0)}(\mathcal{U}\mathfrak{g})$*, and* $p_{\mathcal{P}} : (\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)} \to \widetilde{\mathfrak{g}}^{(1)}_{\mathcal{P}}$ *is the natural morphism of dg-schemes:*

$$\begin{tikzcd} \mathrm{DGCoh}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) \arrow[r, "R(p_{\mathcal{P}})_*"] \arrow[d, "\widehat{\gamma}^{\mathcal{P}}_{\mu}", "\wr"'] & \mathcal{D}^b\mathrm{Coh}_{\mathcal{P}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}_{\mathcal{P}}) \arrow[d, "\gamma^{\mathcal{P}}_{\mu}", "\wr"'] \\ \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{\mu}((\mathcal{U}\mathfrak{g})_0) \arrow[r, "\mathrm{Incl}"] & \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(\mu,0)}(\mathcal{U}\mathfrak{g}). \end{tikzcd}$$

4 Simples correspond to projective covers under $\kappa_{\mathcal{B}}$

In this section we state the result which will be the key of our arguments, Theorem 4.4.3. Before that, we prove several technical results needed for this statement.

4.1 Restricted dominant weights

Consider the element $\tau_0 := t_\rho \cdot w_0$ of W'_{aff}. Recall the formula for the length of an element of W'_{aff}: for $w \in W$ and $x \in \mathbb{X}$ we have (see [IM65, 1.23]):

$$\ell(w \cdot t_x) = \sum_{\substack{\alpha \in R^+, \\ w\alpha \in R^+}} |\langle x, \alpha^\vee \rangle| + \sum_{\substack{\alpha \in R^+, \\ w\alpha \in R^-}} |1 + \langle x, \alpha^\vee \rangle|. \tag{4.1.1}$$

In particular, we obtain $\ell(\tau_0) = \sum_{\alpha \in R^+} (\langle \rho, \alpha^\vee \rangle - 1)$.

Let us define

$$W^0 := \{w \in W'_{\mathrm{aff}} \mid w \bullet C_0 \text{ contains a restricted dominant weight}\}.$$

If $\lambda \in C_0$, W^0 is also the set of $w \in W'_{\mathrm{aff}}$ such that $w \bullet \lambda$ is restricted dominant. It is a finite set, in bijection with W, under our assumption $p > h$ (see *e.g.* the proof of Proposition 4.1.2 below).

Proposition 4.1.2. *The map $w \mapsto \tau_0 w$ is an involution of W^0. Moreover, if $w \in W^0$ we have $\ell(\tau_0 w) = \ell(\tau_0) - \ell(w)$.*

Proof. It is immediate from the definition that $(\tau_0)^2 = 1$. Hence to prove the first assertion it is sufficient to prove that if $w \in W^0$ then $\tau_0 w \in W^0$. As remarked above, we have $W^0 := \{w \in W'_{\mathrm{aff}} \mid w \bullet 0 \text{ is restricted dominant}\}$. Write $w = t_\lambda \cdot v$ with $\lambda \in \mathbb{X}$ and $v \in W$. Then $w \bullet 0 = v(\rho) + p\lambda - \rho$. Hence if $\alpha \in \Phi$ we have $\langle w \bullet 0, \alpha^\vee \rangle = \langle \rho, (v^{-1}\alpha)^\vee \rangle + p\langle \lambda, \alpha^\vee \rangle - 1$. As $p > h$, we have $|\langle \rho, (v^{-1}\alpha)^\vee \rangle| < p$. Hence, $w \bullet 0$ dominant restricted implies:

$$\langle \lambda, \alpha^\vee \rangle = \begin{cases} 0 & \text{if } v^{-1}\alpha \in R^+; \\ 1 & \text{if } v^{-1}\alpha \in R^-. \end{cases} \tag{4.1.3}$$

In both cases, $\langle w \bullet 0, \alpha^\vee \rangle \in \{0, 1, \cdots, p-2\}$.

Now $\tau_0 w \bullet 0 = w_0(w \bullet 0 + \rho) + (p-1)\rho = w_0(w \bullet 0) + (p-2)\rho$. Hence if $\alpha \in \Phi$, $\langle \tau_0 w \bullet 0, \alpha^\vee \rangle = \langle w \bullet 0, (w_0\alpha)^\vee \rangle + (p-2)$. We have $w_0\alpha \in -\Phi$, hence, by the previous remark, $\langle w \bullet 0, (w_0\alpha)^\vee \rangle \in \{-p+2, \cdots, 0\}$. Thus $\tau_0 w \in W^0$, and the first assertion of the proposition follows.

Let us compute $\ell(\tau_0 w)$. We have $\tau_0 w = w_0 v \cdot t_{v^{-1}(\lambda - \rho)}$. Hence, by (4.1.1),

$$\begin{aligned} \ell(\tau_0 w) &= \sum_{\substack{\alpha \in R^+, \\ w_0 v\alpha \in R^+}} |\langle v^{-1}(\lambda - \rho), \alpha^\vee \rangle| + \sum_{\substack{\alpha \in R^+, \\ w_0 v\alpha \in R^-}} |1 + \langle v^{-1}(\lambda - \rho), \alpha^\vee \rangle| \\ &= \sum_{\substack{\alpha \in R^+, \\ v\alpha \in R^-}} |\langle \lambda - \rho, (v\alpha)^\vee \rangle| + \sum_{\substack{\alpha \in R^+, \\ v\alpha \in R^+}} |1 + \langle \lambda - \rho, (v\alpha)^\vee \rangle|. \end{aligned}$$

It follows from (4.1.3) that for $\alpha \in \Phi$ we have $0 \leq \langle \lambda, \alpha^\vee \rangle \leq \langle \rho, \alpha^\vee \rangle$. Hence the same is true for any $\alpha \in R^+$. Moreover, if $v^{-1}\alpha \in R^+$ then the second inequality is strict, and if $v^{-1}\alpha \in R^-$ then the first one is strict. Hence

$$\begin{aligned}\ell(\tau_0 w) &= \sum_{\substack{\alpha \in R^+, \\ v\alpha \in R^-}} \langle \lambda - \rho, (v\alpha)^\vee \rangle + \sum_{\substack{\alpha \in R^+, \\ v\alpha \in R^+}} (-1 + \langle \rho - \lambda, (v\alpha)^\vee \rangle) \\ &= \sum_{\beta \in R^+} \langle \rho, \beta^\vee \rangle + \sum_{\substack{\alpha \in R^+, \\ v\alpha \in R^-}} \langle \lambda, (v\alpha)^\vee \rangle \\ &\quad - \sum_{\substack{\alpha \in R^+, \\ v\alpha \in R^+}} \langle \lambda, (v\alpha)^\vee \rangle - \#\{\alpha \in R^+ \mid v\alpha \in R^+\}.\end{aligned}$$

We deduce that

$$\begin{aligned}\ell(\tau_0 w) &= \ell(\tau_0) + \sum_{\substack{\alpha \in R^+, \\ v\alpha \in R^-}} \langle \lambda, (v\alpha)^\vee \rangle \\ &\quad - \sum_{\substack{\alpha \in R^+, \\ v\alpha \in R^+}} \langle \lambda, (v\alpha)^\vee \rangle + \#\{\alpha \in R^+ \mid v\alpha \in R^-\} \\ &= \ell(\tau_0) - \sum_{\substack{\alpha \in R^+, \\ v\alpha \in R^-}} |1 + \langle \lambda, (v\alpha)^\vee \rangle| - \sum_{\substack{\alpha \in R^+, \\ v\alpha \in R^+}} |\langle \lambda, (v\alpha)^\vee \rangle| \\ &= \ell(\tau_0) - \ell(w).\end{aligned}$$

Here the second equality uses the fact that if $\alpha \in R^+$ and $v\alpha \in R^-$ then $\langle \lambda, (v\alpha)^\vee \rangle + 1 \leq 0$, and the third one uses the equality $w = t_\lambda \cdot v = v \cdot t_{v^{-1}\lambda}$ and formula (4.1.1). This concludes the proof. □

4.2 Coherent sheaves and dg-sheaves on $\widetilde{\mathcal{N}}^{(1)}$

As in subsections 2.3 and 3.1, let us consider the following $\mathbb{G}_\mathbf{m}$-dg-algebras on $\mathcal{B}^{(1)}$, with trivial differential:

$$\begin{aligned}\mathcal{S} &:= \mathrm{S}_{\mathcal{O}_{\mathcal{B}^{(1)}}}(\mathcal{T}_{\mathcal{B}^{(1)}}) \quad \text{with } \mathcal{T}_{\mathcal{B}^{(1)}} \text{ in bidegree } (2, -2), \\ \mathcal{R} &:= \mathrm{S}_{\mathcal{O}_{\mathcal{B}^{(1)}}}(\mathcal{T}_{\mathcal{B}^{(1)}}) \quad \text{with } \mathcal{T}_{\mathcal{B}^{(1)}} \text{ in bidegree } (0, -2),\end{aligned}$$

where $\mathcal{T}_{\mathcal{B}^{(1)}}$ is the tangent bundle to $\mathcal{B}^{(1)}$. We have a "regrading" functor

$$\xi : \mathcal{D}_{\mathbb{G}_\mathbf{m}}(\mathcal{B}^{(1)}, \mathcal{S}) \xrightarrow{\sim} \mathcal{D}_{\mathbb{G}_\mathbf{m}}(\mathcal{B}^{(1)}, \mathcal{R}),$$

defined by $\xi(M)^i_j = M^{i-j}_j$. We also have an equivalence of categories (see (2.3.4)):

$$\phi : \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(\mathcal{B}^{(1)}, \mathcal{R}) \xrightarrow{\sim} \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(\widetilde{\mathcal{N}}^{(1)}).$$

As in (2.3.6) we consider the functor

$$\eta : \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) \to \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(\widetilde{\mathcal{N}}^{(1)})$$

defined as the composition

$$\begin{aligned}\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) := \mathcal{D}^{+,\mathrm{qc},\mathrm{fg}}_{\mathbb{G}_{\mathbf{m}}}(\mathcal{B}^{(1)}, \mathcal{S}) \ &\to\ \mathcal{D}^{\mathrm{qc},\mathrm{fg}}_{\mathbb{G}_{\mathbf{m}}}(\mathcal{B}^{(1)}, \mathcal{S}) \\ &\xrightarrow{\xi}\ \mathcal{D}^{\mathrm{qc},\mathrm{fg}}_{\mathbb{G}_{\mathbf{m}}}(\mathcal{B}^{(1)}, \mathcal{R}) \ \xrightarrow{\phi}\ \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}^{(1)}).\end{aligned}$$

Lemma 4.2.1. *There exists a fully faithful triangulated functor*

$$\zeta : \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \ \to\ \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$$

such that $\eta \circ \zeta$ is the inclusion $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \hookrightarrow \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}^{(1)})$ (see [BMR08, 3.1.7]*).*

Proof. In this proof we consider $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$ as the localization of the homotopy category of the category $\mathcal{C}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$ of bounded complexes of $\mathbb{G}_{\mathbf{m}}$-equivariant coherent sheaves on $\widetilde{\mathcal{N}}^{(1)}$, supported on the zero-section. In particular, any object in this category is bounded for both gradings (the cohomological one and the internal one).

Consider the functor

$$\zeta : \mathcal{C}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \ \to\ \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$$

sending the complex M to the dg-module defined by $\zeta(M)^i_j := M^{i+j}_j$. This functor sends quasi-isomorphisms to isomorphisms. Hence it induces a functor $\zeta : \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \to \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$. It is clear that the functor $\eta \circ \zeta$ is isomorphic to the inclusion of the full subcategory $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$ inside $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}^{(1)})$. Hence ζ is faithful. Now we show that it is full.

Let M and N be two objects of $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$. A morphism $f : \zeta(M) \to \zeta(N)$ in $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$ can be represented by a diagram

$$\zeta(M) \xleftarrow{\mathrm{qis}} P \longrightarrow \zeta(N)$$

with P an object of $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$. Let us fix a positive integer a such that $M_j = N_j = 0$ for $|j| \geq a$. We define the object $P^{[1]}$ of $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$ by $(P^{[1]})_j = P_j$ if $j < a$, $(P^{[1]})_j = 0$ if $j \geq a$. This is a sub-dg-module of P (because $\mathcal{S}$ is concentrated in non-positive internal degrees). Moreover, the inclusion $P^{[1]} \hookrightarrow P$ is a quasi-isomorphism. Next we define the sub-dg-module $P^{[2]}$ of $P^{[1]}$ by $(P^{[2]})_j = (P^{[1]})_j$ if $j \leq -a$, $(P^{[2]})_j = 0$ if $j > -a$, and we denote by $P^{[3]}$ the quotient $P^{[1]}/P^{[2]}$. The morphism $P^{[1]} \to P^{[3]}$ is again a quasi-isomorphism. Moreover, we have the diagram

$$\begin{tikzcd} & P \arrow[dl, "\mathrm{qis}"'] \arrow[dr] & \\ \zeta(M) & P^{[1]} \arrow[l, "\mathrm{qis}"'] \arrow[r] \arrow[u, "\mathrm{qis}"'] \arrow[d, "\mathrm{qis}"] & \zeta(N) \\ & P^{[3]} \arrow[ul, "\mathrm{qis}"] \arrow[ur] & \end{tikzcd}$$

because the morphisms $P^{[1]} \to \zeta(M)$ and $P^{[1]} \to \zeta(N)$ factorize through $P^{[3]}$. Hence, replacing P by $P^{[3]}$, we can assume that P is bounded for the internal grading.

Consider now the object $Q := \xi(P)$ of $\mathcal{C}_{\mathbb{G}_\mathbf{m}}(\mathcal{B}^{(1)}, \mathcal{R})$. It is bounded for the internal grading, bounded below for the cohomological grading, and its cohomology is bounded. Using a truncation functor (which is possible since $\mathcal{R}$ is concentrated in non-positive degrees), there exists an object $Q^{[1]}$ of $\mathcal{C}_{\mathbb{G}_\mathbf{m}}(\mathcal{B}^{(1)}, \mathcal{R})$, bounded for both gradings, and a quasi-isomorphism $Q^{[1]} \xrightarrow{\text{qis}} Q$. Then, consider the object $P^{[4]} := \xi^{-1}(Q^{[1]})$ of $\mathcal{C}_{\mathbb{G}_\mathbf{m}}(\mathcal{B}^{(1)}, \mathcal{S})$. It is bounded for both gradings, and there is a quasi-isomorphism $P^{[4]} \xrightarrow{\text{qis}} P$. Thus we can assume P is bounded for both gradings.

Consider now the morphism

$$\phi^{-1}\eta(f) : \phi^{-1}M \to \phi^{-1}N$$

in $\mathcal{D}^{\text{qc,fg}}_{\mathbb{G}_\mathbf{m}}(\mathcal{B}^{(1)}, \mathcal{R})$. As $\mathcal{D}^b\text{Coh}^{\mathbb{G}_\mathbf{m}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$ is a full subcategory of $\mathcal{D}^b\text{Coh}^{\mathbb{G}_\mathbf{m}}(\widetilde{\mathcal{N}}^{(1)})$, there exists a diagram in $\mathcal{C}_{\mathbb{G}_\mathbf{m}}(\mathcal{B}^{(1)}, \mathcal{R})$:

$$\begin{tikzcd}
 & \phi^{-1}\eta(P) \arrow[dl, "\text{qis}"'] \arrow[dr] & \\
\phi^{-1}M & Q^{[2]} \arrow[l, "\text{qis}"'] \arrow[r] \arrow[u, "\text{qis}"'] \arrow[d, "\text{qis}"] & \phi^{-1}N \\
 & \phi^{-1}Q^{[3]} \arrow[ul, "\text{qis}"] \arrow[ur] &
\end{tikzcd}$$

where $Q^{[2]}$ is an object of $\mathcal{D}^{\text{qc,fg}}_{\mathbb{G}_\mathbf{m}}(\mathcal{B}^{(1)}, \mathcal{R})$, and $Q^{[3]}$ is a bounded complex of $\mathbb{G}_\mathbf{m}$-equivariant coherent sheaves on $\widetilde{\mathcal{N}}^{(1)}$, supported on the zero section. Now, using arguments similar to those used above, we can assume $Q^{[2]}$ is bounded for the internal grading, and bounded below for the cohomological one. It easily follows that f is equal to the image under ζ of the morphism defined by the diagram $M \xleftarrow{\text{qis}} Q^{[3]} \to N$. $\square$

4.3 Translation functors

The translation functors for $\mathcal{U}\mathfrak{g}$-modules are defined in [BMR08, 6.1]. In this subsection we prove, in particular cases sufficient for our purposes, that these translation functors (for $\mathcal{U}\mathfrak{g}$-modules) coincide (on G-modules) with the usual translation functors defined *e.g.* in [Jan03, II.7]. We denote by T_λ^μ the translation functors defined in [BMR08], and by $\hat{T}_\lambda^\mu$ the ones defined in [Jan03]. We also denote by $\text{Mod}^{\text{fd}}_\lambda(G)$ the category of finite dimensional G-modules in the block of λ, for $\lambda \in \mathbb{X}$.

We define

$$\overline{C}_0 := \{\nu \in \mathbb{X} \mid \forall \alpha \in R^+,\ 0 \leq \langle \nu + \rho, \alpha^\vee \rangle \leq p\}.$$

Lemma 4.3.1. *Let $\lambda, \mu \in \overline{C}_0$. Consider the following diagram:*

$$\begin{tikzcd} \mathrm{Mod}^{\mathrm{fd}}_{\lambda}(G) \arrow[r, shift left, "\hat{T}^{\mu}_{\lambda}"] \arrow[d, "\mathrm{For}"] & \mathrm{Mod}^{\mathrm{fd}}_{\mu}(G) \arrow[l, shift left, "\hat{T}^{\lambda}_{\mu}"] \arrow[d, "\mathrm{For}"] \\ \mathrm{Mod}^{\mathrm{fg}}_{(\lambda,0)}(\mathcal{U}\mathfrak{g}) \arrow[r, shift left, "T^{\mu}_{\lambda}"] & \mathrm{Mod}^{\mathrm{fg}}_{(\mu,0)}(\mathcal{U}\mathfrak{g}). \arrow[l, shift left, "T^{\lambda}_{\mu}"] \end{tikzcd}$$

If μ is in the closure of the facet of λ, then $\mathrm{For} \circ \hat{T}^{\mu}_{\lambda} \cong T^{\mu}_{\lambda} \circ \mathrm{For}$. *If λ is regular, and μ is on exactly one wall of $\overline{C}_0$, then* $\mathrm{For} \circ \hat{T}^{\lambda}_{\mu} \cong T^{\lambda}_{\mu} \circ \mathrm{For}$.

Proof. We only prove the first isomorphism (the second one can be obtained similarly). Both translation functors are constructed by tensoring with a module (the same for both functors), and then taking a direct summand. *A priori* the direct summand corresponding to $\hat{T}^{\mu}_{\lambda}$ is smaller than the one corresponding to T^{μ}_{λ}. Hence there exists a natural morphism of functors $\mathrm{For} \circ \hat{T}^{\mu}_{\lambda} \to T^{\mu}_{\lambda} \circ \mathrm{For}$. As these functors are exact, and as the category $\mathrm{Mod}^{\mathrm{fd}}_{\lambda}(G)$ is generated by the induced modules $\mathrm{Ind}_B^G(w \bullet \lambda)$ for $w \in W_{\mathrm{aff}}$ and $w \bullet \lambda$ dominant we only have to prove the result for these modules. But the images under our functors of these modules are explicitly known (see [Jan03, II.7.11 and II.7.12] and [BMR06, 2.2.3]), and they indeed coincide. □

From now on, for simplicity we do not write the functors "For". It follows from this lemma that the usual rules for computing the images of simple or induced modules under translation functors (see [Jan03, II.7]) generalize. For instance, if μ is in the closure of the facet of λ (both in $\overline{C}_0$), then

$$T^{\mu}_{\lambda}\mathrm{Ind}_B^G(w \bullet \lambda) = \mathrm{Ind}_B^G(w \bullet \mu)$$

for any $w \in W'_{\mathrm{aff}}$. If moreover $w \bullet \lambda$ is dominant and restricted, then

$$T^{\mu}_{\lambda}L(w \bullet \lambda) = \begin{cases} L(w \bullet \mu) & \text{if } w \bullet \mu \text{ is in the upper closure} \\ & \text{of the facet of } w \bullet \lambda; \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.2}$$

To finish this subsection, let us remark that, as the tensor product of two restricted $\mathcal{U}\mathfrak{g}$-modules is again restricted, for λ, μ in $\mathbb{X}$ the functor $T^{\mu}_{\lambda} : \mathrm{Mod}^{\mathrm{fg}}_{(\lambda,0)}(\mathcal{U}\mathfrak{g}) \to \mathrm{Mod}^{\mathrm{fg}}_{(\mu,0)}(\mathcal{U}\mathfrak{g})$ induces a functor denoted similarly:

$$T^{\mu}_{\lambda} : \mathrm{Mod}^{\mathrm{fg}}_{\lambda}((\mathcal{U}\mathfrak{g})_0) \ \to \ \mathrm{Mod}^{\mathrm{fg}}_{\mu}((\mathcal{U}\mathfrak{g})_0).$$

4.4 Objects corresponding to simple and projective modules

Let $\lambda \in C_0$ be arbitrary. Recall that, by a theorem of Curtis (see [Cur60]), a complete system of (non isomorphic) simple $(\mathcal{U}\mathfrak{g})_0$-modules is given by the restriction to $(\mathcal{U}\mathfrak{g})_0$ of the simple G-modules $L(\nu)$ for $\nu \in \mathbb{X}$ restricted and dominant. The simple objects in the category $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^\lambda)$ (or similarly in the category $\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$), *i.e.* the simple $(\mathcal{U}\mathfrak{g})_0$-modules with Harish-Chandra central character λ are the $L(w \bullet \lambda)$, for $w \in W'_{\mathrm{aff}}$ such that $w \bullet \lambda$ is restricted and dominant, *i.e.* for $w \in W^0$ (see subsection 4.1).

Recall the equivalence $\epsilon_\lambda^{\mathcal{B}}$ of (1.2.3) in chapter I. For $w \in W^0$ we define

$$\mathcal{L}_w := (\epsilon_\lambda^{\mathcal{B}})^{-1} L(w \bullet \lambda) \quad \in \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \tag{4.4.1}$$

This object does not depend on the choice of $\lambda \in C_0$. Indeed, let μ be another weight in C_0. By [BMR08, 6.1.2.(a)], for any $\mathcal{F} \in \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)})$ we have

$$T_\lambda^\mu \gamma_\lambda^{\mathcal{B}}(\mathcal{F}) \cong R\Gamma\big(\mathcal{O}_{\mathcal{B}}(\mu - \lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} (\mathcal{M}^\lambda \otimes_{\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}} \mathcal{F})\big)$$

(in this formula, $\mathcal{M}^\lambda \otimes_{\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}} \mathcal{F}$ is considered as a sheaf of $\widetilde{\mathcal{D}}$-modules on $\mathcal{B}$). By our choice of splitting bundles (see [BMR06, 1.3.5]), we have $\mathcal{M}^\mu = \mathcal{O}_{\mathcal{B}}(\mu - \lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{M}^\lambda$, hence

$$T_\lambda^\mu \circ \gamma_\lambda^{\mathcal{B}}(\mathcal{F}) \cong \gamma_\mu^{\mathcal{B}}(\mathcal{F}).$$

Similarly, T_λ^μ restricts to a functor $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^\lambda) \to \mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^\mu)$, and for an object $\mathcal{F} \in \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$ we have

$$T_\lambda^\mu \circ \epsilon_\lambda^{\mathcal{B}}(\mathcal{F}) \cong \epsilon_\mu^{\mathcal{B}}(\mathcal{F}).$$

Hence if $\mathcal{L}_w$ is defined using the weight λ, we have $\epsilon_\mu^{\mathcal{B}}(\mathcal{L}_w) \cong T_\lambda^\mu \circ \epsilon_\lambda^{\mathcal{B}}(\mathcal{L}_w) \cong T_\lambda^\mu L(w \bullet \lambda) \cong L(w \bullet \mu)$, which proves the claim. Here the last isomorphism follows from (4.3.2).

Consider now the category $\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$. The algebra $(\mathcal{U}\mathfrak{g})_0$ is finite dimensional. Hence, if $\mathfrak{Z}_{\mathrm{HC}}^\lambda$ denotes the image in $(\mathcal{U}\mathfrak{g})_0$ of the maximal ideal of $\mathfrak{Z}_{\mathrm{HC}} \cong \mathrm{S}(\mathfrak{h})^{(W,\bullet)}$ corresponding to the character induced by λ, the sequence of ideals of $(\mathcal{U}\mathfrak{g})_0$

$$\langle \mathfrak{Z}_{\mathrm{HC}}^\lambda \rangle \;\supset\; \langle \mathfrak{Z}_{\mathrm{HC}}^\lambda \rangle^2 \;\supset\; \langle \mathfrak{Z}_{\mathrm{HC}}^\lambda \rangle^3 \;\supset\; \dots$$

stabilizes. Thus, for n sufficiently large, the category $\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$ is equivalent to the category of finitely generated modules over $(\mathcal{U}\mathfrak{g})_0 / \langle \mathfrak{Z}_{\mathrm{HC}}^\lambda \rangle^n$. We denote this algebra by $((\mathcal{U}\mathfrak{g})_0)^{\hat{\lambda}}$, or simply $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$.

As seen above, the simple $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$-modules are the $L(w \bullet \lambda)$ for $w \in W^0$. We denote by $P(w \bullet \lambda)$ the projective cover of $L(w \bullet \lambda)$ in the category of $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$-modules. Recall the equivalence $\widehat{\gamma}_\lambda^{\mathcal{B}}$ of Theorem 3.3.3. For $w \in W^0$ we define

$$\mathcal{P}_w := (\widehat{\gamma}_\lambda^{\mathcal{B}})^{-1} P(w \bullet \lambda) \quad \in \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}). \tag{4.4.2}$$

As above, this object does not depend on the choice of $\lambda \in C_0$.

Our key-result states that the objects $\mathcal{L}_w$ and $\mathcal{P}_w$ correspond under the linear Koszul duality $\kappa_{\mathcal{B}}$ of (3.1.1). More precisely, consider the forgetful functor $\mathrm{For} : \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}^{(1)}) \to \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$. If $\mathcal{G} \in \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$, we say that an object $\mathcal{F}$ of $\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}^{(1)})$ is a *lift* of $\mathcal{G}$ if $\mathrm{For}(\mathcal{F}) \cong \mathcal{G}$. We use the same terminology for objects in the categories $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$ and $\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$. In section 8 we will prove the following result.

Theorem 4.4.3. *Assume $p > h$ is large enough so that Lusztig's conjecture is true*[5].

There is a unique choice of lifts $\mathcal{P}_v^{\mathrm{gr}} \in \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$ *of* $\mathcal{P}_v$*, resp.* $\mathcal{L}_v^{\mathrm{gr}} \in \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$ *of* $\mathcal{L}_v$ *($v \in W^0$), such that for all* $w \in W^0$ *we have in* $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$*:*

$$\kappa_{\mathcal{B}}^{-1} \mathcal{P}^{\mathrm{gr}}_{\tau_0 w} \ \cong \ \zeta(\mathcal{L}_w^{\mathrm{gr}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho).$$

The unicity statement in this theorem is not difficult to prove (see 8.1). The existence is much more complex. To prove it we will need several tools, which we introduce in sections 5, 6 and 7.

As explained above, this statement does not depend on the choice of a weight $\lambda \in C_0$. From now on, for simplicity we mainly restrict to the case $\lambda = 0$.

5 Braid group actions and translation functors

In this section we introduce important technical tools for our study: the (geometric) braid group actions and the geometric counterparts of the translation functors.

5.1 Braid group actions

In this subsection we recall the main results of chapter II. Denote by Φ_{aff} the set which contains Φ and additional symbols for each element of $S_{\mathrm{aff}} - S$. If $\alpha_0 \in \Phi_{\mathrm{aff}} - \Phi$, we denote by s_{α_0} the corresponding element of $S_{\mathrm{aff}} - S$. The elements of $\Phi_{\mathrm{aff}} - \Phi$ are called *affine simple roots*, and the ones of Φ *finite simple roots*.

We use the same notation as in chapter II for the extended affine braid group B'_{aff} (see II.1.1), the convolution functors (see II.2.1), and the varieties S_α and S'_α (see II.2.3). In Theorem II.2.3.2 we have proved that there exists an action of B'_{aff} on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})$ (respectively $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)})$) for which:

(i) For $x \in \mathbb{X}$, the action of θ_x is given by the convolution with kernel $\Delta_*(\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}(x))$ (respectively $\Delta_*(\mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}(x))$), where Δ is the diagonal embedding;

(ii) For $\alpha \in \Phi$, the action of T_α is given by the convolution with kernel $\mathcal{O}_{S_\alpha^{(1)}}$ (respectively $\mathcal{O}_{S'_\alpha{}^{(1)}}$). The action of $(T_\alpha)^{-1}$ is the convolution with kernel $\mathcal{O}_{S_\alpha^{(1)}}(-\rho, \rho - \alpha)$ (respectively $\mathcal{O}_{S'_\alpha{}^{(1)}}(-\rho, \rho - \alpha)$).

Moreover, the actions on $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)})$ and $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})$ correspond under the direct image functor $i_* : \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)}) \to \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})$ where i is the closed embedding $\widetilde{\mathcal{N}}^{(1)} \hookrightarrow \widetilde{\mathfrak{g}}^{(1)}$.

In [BMR06] the authors have constructed an action of B'_{aff} on $\mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(0,0)}(\mathcal{U}\mathfrak{g})$ (see II.6.3); for $b \in B'_{\mathrm{aff}}$, let us denote by

$$\mathbf{I}_b : \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(0,0)}(\mathcal{U}\mathfrak{g}) \to \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(0,0)}(\mathcal{U}\mathfrak{g})$$

[5] See 0.5 for comments.

the corresponding action. On the other hand, let us denote by

$$\mathbf{J}_b : \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)}) \to \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)}),$$
$$\text{resp.} \qquad \mathbf{K}_b : \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)}) \to \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)}),$$

the actions of b given by Theorem II.2.3.2. We have proved in subsection II.6.3 that for any $b \in B'_{\mathrm{aff}}$ the following diagram is commutative:

$$\begin{tikzcd} \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}) \arrow[r, "\mathbf{J}_b"] \arrow[d, "\gamma_0^{\mathcal{B}}"', "\wr"] & \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}) \arrow[d, "\wr"', "\gamma_0^{\mathcal{B}}"] \\ \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(0,0)}(\mathcal{U}\mathfrak{g}) \arrow[r, "\mathbf{I}_b"] & \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(0,0)}(\mathcal{U}\mathfrak{g}). \end{tikzcd} \tag{5.1.1}$$

Let us point out that our notations are not exactly the same as in subsection II.6.3.

5.2 Graded versions of the actions

Let us define actions of $\mathbb{G}_{\mathbf{m}} \cong \Bbbk^\times$ on $\widetilde{\mathfrak{g}}^{(1)}$ and $\widetilde{\mathcal{N}}^{(1)}$, by setting

$$t \cdot (X, gB) = (t^{-2} \cdot X, gB), \quad \text{resp.} \quad t \cdot (X, gB) = (t^2 \cdot X, gB) \tag{5.2.1}$$

for $t \in \Bbbk^\times$ and (X, gB) in $\widetilde{\mathfrak{g}}^{(1)}$, respectively $\widetilde{\mathcal{N}}^{(1)}$. Note that the action on $\widetilde{\mathcal{N}}^{(1)}$ is *not* the restriction of the action on $\widetilde{\mathfrak{g}}^{(1)}$, but rather the *dual* action. This is consistent with the constructions of subsection 3.1. Recall also that the action of $\Bbbk$ on $\mathfrak{g}^{*(1)}$ is twisted: if $\mathrm{Fr} : \mathfrak{g}^* \to \mathfrak{g}^{*(1)}$ denotes the Frobenius morphism, and if $t \in \Bbbk$, then we have $t \cdot \mathrm{Fr}(X) = \mathrm{Fr}(t^{1/p}X)$. As in subsection 2.5, we denote by $\langle 1 \rangle$ the shift in the grading given by the tensor product with the one-dimensional $\mathbb{G}_{\mathbf{m}}$-module given by $\mathrm{Id}_{\mathbb{G}_{\mathbf{m}}}$. An easy extension of the constructions of chapter II yields:

Proposition 5.2.2. *There exists an action of B'_{aff} on the category $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathfrak{g}}^{(1)})$ (resp. $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}^{(1)})$) for which:*

(i) *For $x \in \mathbb{X}$, the action of θ_x is given by the convolution with kernel $\Delta_*\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}(x)$ (respectively $\Delta_*\mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}(x)$), where Δ is the diagonal embedding;*

(ii) *For $\alpha \in \Phi$, the action of T_α is given by the convolution with kernel $\mathcal{O}_{S_\alpha^{(1)}}\langle -1 \rangle$ (respectively $\mathcal{O}_{S'^{(1)}_\alpha}\langle 1 \rangle$). Moreover, the action of $(T_\alpha)^{-1}$ is the convolution with kernel $\mathcal{O}_{S_\alpha^{(1)}}(-\rho, \rho - \alpha)\langle -1 \rangle$ (respectively $\mathcal{O}_{S'^{(1)}_\alpha}(-\rho, \rho - \alpha)\langle 1 \rangle$).*

Proof. Here we only consider $\widetilde{\mathfrak{g}}^{(1)}$ (the proof for $\widetilde{\mathcal{N}}^{(1)}$ is similar). All we have to do is to observe that the varieties S_α are $\mathbb{G}_{\mathbf{m}}$-stable subvarieties of $\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$, and that all the constructions and proofs of chapter II respect the $\mathbb{G}_{\mathbf{m}}$-equivariant structure. The only subtlety concerns the proof of Proposition II.2.4.2 (see also section II.8). In this proof, the $\mathbb{G}_{\mathbf{m}}$-equivariant version of the exact sequence $\mathcal{O}_{V_\alpha^1} \hookrightarrow \mathcal{O}_{V_\alpha}(\rho - \alpha, -\rho, 0) \twoheadrightarrow \mathcal{O}_{V_\alpha^2}(\rho - \alpha, -\rho, 0)$ is

$$\mathcal{O}_{V_\alpha^1}\langle 2 \rangle \hookrightarrow \mathcal{O}_{V_\alpha}(\rho - \alpha, -\rho, 0) \twoheadrightarrow \mathcal{O}_{V_\alpha^2}(\rho - \alpha, -\rho, 0).$$

The rest of the proof works similarly. $\square$

Now we consider the dg-scheme $(\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}$. Recall the notation for categories of dg-modules in section 1. By definition (see equation (2.3.9)),

$$\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \;\cong\; \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_{\mathbf{m}}}(\mathcal{B}^{(1)},\, \Lambda_{\mathcal{O}_{\mathcal{B}^{(1)}}}(\mathcal{T}^{\vee}_{\mathcal{B}^{(1)}})).$$

This realization was constructed using the resolution

$$(\mathrm{S}(\mathfrak{g}^{(1)}) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \Lambda_{\mathcal{O}_{\mathcal{B}^{(1)}}}(\mathcal{T}^{\vee}_{\mathcal{B}^{(1)}}) \xrightarrow{\mathrm{qis}} \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}},$$

where $\pi : \widetilde{\mathfrak{g}}^{(1)} \to \mathcal{B}^{(1)}$ denotes the projection to the base. Consider also the Koszul resolution

$$(\mathrm{S}(\mathfrak{g}^{(1)}) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}}) \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}) \xrightarrow{\mathrm{qis}} \mathcal{O}_{\mathcal{B}^{(1)}}.$$

There exist quasi-isomorphisms of dg-algebras on $\mathcal{B}^{(1)}$:

$$\begin{aligned}\big(\Lambda_{\mathcal{O}_{\mathcal{B}^{(1)}}}(\mathcal{T}^{\vee}_{\mathcal{B}^{(1)}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} (\mathrm{S}(\mathfrak{g}^{(1)}) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}})\big) \otimes_{\mathrm{S}(\mathfrak{g}^{(1)}) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}}} \big((\mathrm{S}(\mathfrak{g}^{(1)}) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}}) \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})\big) \\ \xrightarrow{\mathrm{qis}} \big(\Lambda_{\mathcal{O}_{\mathcal{B}^{(1)}}}(\mathcal{T}^{\vee}_{\mathcal{B}^{(1)}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} (\mathrm{S}(\mathfrak{g}^{(1)}) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}})\big) \otimes_{\mathrm{S}(\mathfrak{g}^{(1)}) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}} \cong \Lambda_{\mathcal{O}_{\mathcal{B}^{(1)}}}(\mathcal{T}^{\vee}_{\mathcal{B}^{(1)}})\end{aligned}$$

and

$$\begin{aligned}\big(\Lambda_{\mathcal{O}_{\mathcal{B}^{(1)}}}(\mathcal{T}^{\vee}_{\mathcal{B}^{(1)}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} (\mathrm{S}(\mathfrak{g}^{(1)}) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}})\big) \otimes_{\mathrm{S}(\mathfrak{g}^{(1)}) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}}} \big((\mathrm{S}(\mathfrak{g}^{(1)}) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}}) \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})\big) \\ \xrightarrow{\mathrm{qis}} \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\mathrm{S}(\mathfrak{g}^{(1)}) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}}} \big((\mathrm{S}(\mathfrak{g}^{(1)}) \otimes_{\Bbbk} \mathcal{O}_{\mathcal{B}^{(1)}}) \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})\big) \cong \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}).\end{aligned}$$

Using Proposition 1.5.6, and a $\mathbb{G}_{\mathbf{m}}$-equivariant analogue, we deduce:

Lemma 5.2.3. *There exist equivalences of categories*

$$\begin{aligned}\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) &\cong \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_{\mathbf{m}}}(\mathcal{B}^{(1)},\, (\pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}) \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})), \\ \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) &\cong \mathcal{D}^{\mathrm{qc,fg}}(\mathcal{B}^{(1)},\, (\pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}) \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})),\end{aligned}$$

where $(\pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}) \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})$ is considered as a dg-algebra equipped with a Koszul differential, with $\pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}$ in cohomological degree 0 and $\mathfrak{g}^{(1)}$ in cohomological degree -1. In the first equivalence, the internal grading on $\pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}$ is induced by the $\mathbb{G}_{\mathbf{m}}$-action (5.2.1), and $\mathfrak{g}^{(1)}$ is in bidegree $(-1, 2)$.

Recall that $p : (\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)} \to \widetilde{\mathfrak{g}}^{(1)}$ denotes the natural morphism of dg-schemes.

Proposition 5.2.4. *There exist actions of B'_{aff} on the categories* $\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$ *and* $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$ *such that the functors*

$$\begin{tikzcd} \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[r, "\mathrm{For}"] \arrow[d, "R(p_{\mathbb{G}_{\mathbf{m}}})_*"'] & \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[d, "Rp_*"] \\ \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathfrak{g}}^{(1)}) \arrow[r, "\mathrm{For}"] & \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)}) \end{tikzcd}$$

commute with the action of B'_{aff}.

Proof. We give the proof for the category $\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$ (the $\mathbb{G}_{\mathbf{m}}$-equivariant case is similar). As above, let $\pi : \widetilde{\mathfrak{g}}^{(1)} \to \mathcal{B}^{(1)}$ be the natural morphism. We denote by $p_i : \widetilde{\mathfrak{g}}^{(1)} \times \widetilde{\mathfrak{g}}^{(1)} \to \widetilde{\mathfrak{g}}^{(1)}$, $q_i : \mathcal{B}^{(1)} \times \mathcal{B}^{(1)} \to \mathcal{B}^{(1)}$ the natural projections ($i = 1, 2$). Recall that π is affine, hence the functor π_* is an equivalence of categories between $\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})$ and $\mathrm{Coh}(\mathcal{B}^{(1)}, \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}})$ (see [Gro61a, 1.4.3]; see also Lemma 2.3.2 and its proof).

If $\mathcal{F}$ is in $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})$, by [Gro61a, 1.5.7.1] we have

$$(\pi \times \pi)_* (p_1^* \mathcal{F}) \cong \big((\pi \times \pi)_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)} \times \widetilde{\mathfrak{g}}^{(1)}} \big) \otimes_{q_1^* \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}} q_1^* \pi_* \mathcal{F}.$$

Using [Gro61a, 1.4.8.1], it follows that if $\alpha \in \Phi$,

$$(\pi \times \pi)_* (p_1^* \mathcal{F} \overset{L}{\otimes}_{\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)} \times \widetilde{\mathfrak{g}}^{(1)}}} \mathcal{O}_{S_\alpha^{(1)}}) \cong \big((\pi \times \pi)_* \mathcal{O}_{S_\alpha^{(1)}} \big) \overset{L}{\otimes}_{q_1^* \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}} q_1^* \pi_* \mathcal{F}.$$

Hence, finally,

$$\begin{array}{rcl} \pi_* F^{\mathcal{O}_{S_\alpha^{(1)}}}_{\widetilde{\mathfrak{g}}^{(1)} \to \widetilde{\mathfrak{g}}^{(1)}}(\mathcal{F}) & \cong & R(q_2)_* (\pi \times \pi)_* (p_1^* \mathcal{F} \overset{L}{\otimes}_{\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)} \times \widetilde{\mathfrak{g}}^{(1)}}} \mathcal{O}_{S_\alpha^{(1)}}) \\ & \cong & R(q_2)_* ((\pi \times \pi)_* \mathcal{O}_{S_\alpha^{(1)}} \overset{L}{\otimes}_{q_1^* \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}} q_1^* \pi_* \mathcal{F}). \end{array} \tag{5.2.5}$$

Moreover, these isomorphisms are functorial. In this formula, $(\pi \times \pi)_* \mathcal{O}_{S_\alpha^{(1)}}$ is considered as a right $q_1^* \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}$-module, and a left $q_2^* \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}$-module. Formula (5.2.5) has a natural dg-version, which will give the definition of the action of T_α.

We define the action of B'_{aff} using the equivalences of categories of Lemma 5.2.3. It is enough (as in Theorem II.2.3.2 and Proposition 5.2.2) to define the action of the generators θ_x ($x \in \mathbb{X}$) and T_α ($\alpha \in \Phi$), and to prove that they satisfy the relations of Theorem II.1.1.3.

First, if $x \in \mathbb{X}$ the action of θ_x is defined as the tensor product with the line bundle $\mathcal{O}_{\mathcal{B}^{(1)}}(x)$. Let $\alpha \in \Phi$. Consider the functor

$$\left\{ \begin{array}{ccc} \mathcal{C}(\mathcal{B}^{(1)}, \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) & \to & \mathcal{C}(\mathcal{B}^{(1)} \times \mathcal{B}^{(1)}, q_2^*(\pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))) \\ \mathcal{G} & \mapsto & \big((\pi \times \pi)_* \mathcal{O}_{S_\alpha^{(1)}} \big) \otimes_{q_1^* \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}} q_1^* \mathcal{G} \end{array} \right.$$

where $(\pi \times \pi)_* \mathcal{O}_{S_\alpha^{(1)}}$ is considered as a bimodule, as above. This functor has a left derived functor (which can be computed using left K-flat resolutions), denoted by

$$\mathcal{G} \mapsto (\pi \times \pi)_* \mathcal{O}_{S_\alpha^{(1)}} \overset{L}{\otimes}_{q_1^* \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}} q_1^* \mathcal{G}.$$

Let

$$\widetilde{q}_2 : \mathcal{C}(\mathcal{B}^{(1)} \times \mathcal{B}^{(1)}, q_2^*(\pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))) \to \mathcal{C}(\mathcal{B}^{(1)}, \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$$

be the natural morphism, induced by q_2. Then we define the action of T_α as the functor

$$F_\alpha : \mathcal{G} \mapsto R(\widetilde{q}_2)_* \big((\pi \times \pi)_* \mathcal{O}_{S_\alpha^{(1)}} \overset{L}{\otimes}_{q_1^* \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}} q_1^* \mathcal{G} \big).$$

Easy arguments show that this functor indeed restricts to the subcategories of dg-modules with quasi-coherent, locally finitely generated cohomology. Moreover, the following diagram commutes:

$$\begin{tikzcd} \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[r, "F_\alpha"] \arrow[d, "Rp_*"'] & \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[d, "Rp_*"] \\ \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)}) \arrow[r, "\mathbf{J}_{T_\alpha}"] & \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)}) \end{tikzcd}$$

(see the remarks at the beginning of this proof, and use the fact that a K-flat $\pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})$-dg-module is also K-flat over $\pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}$).

With these definitions, it follows easily from the results of chapter II that the actions of the T_α's and the θ_x's satisfy the relations of the definition of B'_{aff}. □

For $b \in B'_{\mathrm{aff}}$, we let

$$\begin{aligned} \mathbf{J}_b^{\mathbb{G}_\mathbf{m}} : \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(\widetilde{\mathfrak{g}}^{(1)}) &\to \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(\widetilde{\mathfrak{g}}^{(1)}), \\ \mathbf{K}_b^{\mathbb{G}_\mathbf{m}} : \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(\widetilde{\mathcal{N}}^{(1)}) &\to \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(\widetilde{\mathcal{N}}^{(1)}), \\ \mathbf{J}_b^{\mathrm{dg}} : \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) &\to \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}), \\ \mathbf{J}_b^{\mathrm{dg,gr}} : \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) &\to \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \end{aligned}$$

denote the action of b given by Propositions 5.2.2 and 5.2.4.

It follows[6] in particular from Proposition 5.2.4 that the B'_{aff}-action on $\mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(0,0)}(\mathcal{U}\mathfrak{g})$ factorizes through an action on $\mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_0((\mathcal{U}\mathfrak{g})_0)$, which corresponds to the action on the category $\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$ via the equivalence $\widehat{\gamma}_0^{\mathcal{B}}$ of Theorem 3.3.3. We denote by

$$\mathbf{I}_b^{\mathrm{res}} : \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_0((\mathcal{U}\mathfrak{g})_0) \to \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_0((\mathcal{U}\mathfrak{g})_0)$$

the action of $b \in B'_{\mathrm{aff}}$.

5.3 Some exact sequences

In this subsection we recall some exact sequences constructed in chapter II. Consider the subvariety $S'_\alpha \subset \widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}$. Geometrically, it can be described as:

$$S'_\alpha = \{(X, g_1 B, g_2 B) \in \mathfrak{g}^* \times \mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B} \mid X_{|g_1 \cdot \mathfrak{b} + g_2 \cdot \mathfrak{b}} = 0\}.$$

It has two irreducible components. One is $\Delta\widetilde{\mathcal{N}}$, the diagonal embedding of $\widetilde{\mathcal{N}}$, and the other one is

$$Y_\alpha := \{(X, g_1 B, g_2 B) \in \mathfrak{g}^* \times (\mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B}) \mid X_{|g_1 \cdot \mathfrak{p}_\alpha} = 0\},$$

which is a vector bundle on $\mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B}$, of rank $\dim(\mathfrak{g}/\mathfrak{b}) - 1$.

[6] Of course, the first assertion can also be proved directly.

Recall the morphism $\widetilde{\pi}_\alpha : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}_\alpha$ (see (1.1.1) in chapter I). There exist exact sequences of quasi-coherent sheaves on $\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$, resp. $\widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}$ (see Corollary II.6.2.2 and Lemma II.7.1.1):

$$\mathcal{O}_{\Delta\widetilde{\mathfrak{g}}} \hookrightarrow \mathcal{O}_{\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}} \twoheadrightarrow \mathcal{O}_{S_\alpha}, \tag{5.3.1}$$

$$\mathcal{O}_{S_\alpha}(\rho - \alpha, -\rho) \hookrightarrow \mathcal{O}_{\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}} \twoheadrightarrow \mathcal{O}_{\Delta\widetilde{\mathfrak{g}}}, \tag{5.3.2}$$

$$\mathcal{O}_{\Delta\widetilde{\mathcal{N}}} \hookrightarrow \mathcal{O}_{S'_\alpha}(\rho - \alpha, -\rho) \twoheadrightarrow \mathcal{O}_{Y_\alpha}(\rho - \alpha, -\rho), \tag{5.3.3}$$

$$\mathcal{O}_{Y_\alpha}(\rho - \alpha, -\rho) \hookrightarrow \mathcal{O}_{S'_\alpha} \twoheadrightarrow \mathcal{O}_{\Delta\widetilde{\mathcal{N}}}. \tag{5.3.4}$$

The exact sequences (5.3.2) and (5.3.4) are $\mathbb{G}_{\mathbf{m}}$-equivariant. The exact sequences (5.3.1) and (5.3.3) admit the $\mathbb{G}_{\mathbf{m}}$-equivariant analogues

$$\mathcal{O}_{\Delta\widetilde{\mathfrak{g}}}\langle 2 \rangle \hookrightarrow \mathcal{O}_{\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}} \twoheadrightarrow \mathcal{O}_{S_\alpha}, \tag{5.3.5}$$

$$\mathcal{O}_{\Delta\widetilde{\mathcal{N}}}\langle -2 \rangle \hookrightarrow \mathcal{O}_{S'_\alpha}(\rho - \alpha, -\rho) \twoheadrightarrow \mathcal{O}_{Y_\alpha}(\rho - \alpha, -\rho). \tag{5.3.6}$$

Remark 5.3.7. We have $\mathcal{O}_{\mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B}}(\rho - \alpha, -\rho) \cong \mathcal{O}_{\mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B}}(-\rho, \rho - \alpha)$ (see subsection II.2.4). Hence we can exchange $-\rho$ and $\rho - \alpha$ in these exact sequences.

5.4 Geometric counterparts of the translation functors

Let us recall the geometric interpretation of the translation functors given in [BMR06] (see subsection I.1.3). Let P be a parabolic subgroup of G containing B and let $\mathcal{P} = G/P$. Recall the morphism $\widetilde{\pi}_{\mathcal{P}}$ of (1.1.1) in chapter I. Let λ and μ be as in Proposition I.1.3.1. The morphism $\widetilde{\pi}_{\mathcal{P}} : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}_{\mathcal{P}}$ induces a morphism of dg-schemes

$$\widehat{\pi}_{\mathcal{P}} : (\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)} \to (\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}. \tag{5.4.1}$$

This morphism can be realized in two equivalent ways: either as the morphism of dg-ringed spaces $(\widetilde{\mathfrak{g}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \to (\widetilde{\mathfrak{g}}_{\mathcal{P}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{P}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$, or as the morphism of dg-ringed spaces $(\mathcal{B}^{(1)}, \Lambda_{\mathcal{O}_{\mathcal{B}^{(1)}}}(\mathcal{T}^\vee_{\mathcal{B}^{(1)}})) \to (\mathcal{P}^{(1)}, \Lambda_{\mathcal{O}_{\mathcal{P}^{(1)}}}(\mathcal{T}^\vee_{\mathcal{P}^{(1)}}))$. Easy arguments show that $R(\widehat{\pi}_{\mathcal{P}})_*$ and $L(\widehat{\pi}_{\mathcal{P}})^*$ restrict to functors between the categories $\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$ and $\mathrm{DGCoh}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)})$, with usual compatibility conditions.

Recall the equivalences of Theorems 3.3.3 and 3.3.15. A proof entirely similar to that of [BMR06, 2.2.5] gives also:

Proposition 5.4.2. *Let $\lambda, \mu, P, \mathcal{P}$ be as in Proposition* I.1.3.1. *There exist isomorphisms of functors*

$$T_\lambda^\mu \circ \widehat{\gamma}_\lambda^{\mathcal{B}} \cong \widehat{\gamma}_\mu^{\mathcal{P}} \circ R(\widehat{\pi}_{\mathcal{P}})_* \quad \text{and} \quad T_\mu^\lambda \circ \widehat{\gamma}_\mu^{\mathcal{P}} \cong \widehat{\gamma}_\lambda^{\mathcal{B}} \circ L(\widehat{\pi}_{\mathcal{P}})^*.$$

If $\mathcal{P} = \mathcal{P}_\alpha$ for a finite simple root $\alpha \in \Phi$, we simplify the notation and set $\widehat{\pi}_\alpha := \widehat{\pi}_{\mathcal{P}_\alpha}$.

5.5 Some results from representation theory

One of our main tools will be the reflection functors, defined in the following way.

Definition 5.5.1. Let $\delta \in \Phi_{\mathrm{aff}}$. Let us choose a weight $\mu_\delta \in \mathbb{X}$ which is on the δ-wall of $\overline{C}_0$, and not on any other wall. Then the *reflection functor* R_δ is defined as the composition

$$R_\delta := T^0_{\mu_\delta} \circ T^{\mu_\delta}_0.$$

This functor does not depend on the choice of μ_δ by [BMR06, 2.2.7]. It is an auto-adjoint endofunctor of $\mathrm{Mod}^{\mathrm{fg}}_{(0,0)}(\mathcal{U}\mathfrak{g})$, which stabilizes the subcategory $\mathrm{Mod}^{\mathrm{fg}}_0((\mathcal{U}\mathfrak{g})_0)$. Note that these notations are compatible with those of II.6.3.

In this subsection we recall some classical results describing the action of the reflection functors on simple and projective modules.

Recall that it has been proved that Lusztig's conjecture on the characters of simple G-modules ([Lus80b]) is satisfied for p large enough, with no explicit bound (see 0.5 for details). From now on we make the following assumption:

(#) p is large enough so that Lusztig's conjecture is satisfied.

This restriction is needed only to apply Theorem 5.5.3(i) below.

Let $\delta \in \Phi_{\mathrm{aff}}$. Consider a simple $(\mathcal{U}\mathfrak{g})_0$-module $L(w \bullet 0)$ ($w \in W^0$), where $ws_\delta \bullet 0 > w \bullet 0$ (see subsection 4.4). There are natural morphisms, induced by adjunction,

$$L(w \bullet 0) \xrightarrow{\phi^w_\delta} R_\delta L(w \bullet 0) \xrightarrow{\psi^w_\delta} L(w \bullet 0).$$

It is known (see [Jan03, II.7.20]) that ϕ^w_δ is injective, and that ψ^w_δ is surjective. Let us consider the $\mathcal{U}\mathfrak{g}$-module

$$Q_\delta(w) := \mathrm{Ker}(\psi^w_\delta)/\mathrm{Im}(\phi^w_\delta). \tag{5.5.2}$$

Point (i) of the following theorem is a consequence of a conjecture by Andersen ([And86]), which is known to be equivalent to Lusztig's conjecture on the characters of simple G-modules (see [And86], [Jan03, II.C]). Hence it is true under our hypothesis (#).

Theorem 5.5.3. (i) *Let $\delta \in \Phi_{\mathrm{aff}}$. Let $w \in W^0$ such that $w \bullet 0 < ws_\delta \bullet 0$. Then $Q_\delta(w)$ is a semi-simple $\mathcal{U}\mathfrak{g}$-module.*

(ii) *The simple factors of $Q_\delta(w)$ as a $\mathcal{U}\mathfrak{g}$-module are of the form $L(x \bullet 0)$ for some $x \in W^0$ satisfying $\ell(x) < \ell(ws_\delta)$; plus $L(ws_\delta \bullet 0)$ with multiplicity one if $ws_\delta \in W^0$.*

Proof of (ii). By [Jan03, II.7.19-20] and the strong linkage principle (see [Jan03, II.6.13]), we know that the simple factors of $Q_\delta(w)$ as a G-module are $L(ws_\delta \bullet 0)$ with multiplicity one, and some $L(x \bullet 0)$ with $x \in W_{\mathrm{aff}} - \{w, ws_\delta\}$, such that $x \bullet 0$ is dominant and $x \bullet 0 \uparrow ws_\delta \bullet 0$ (with the notation of [Jan03, II.6.4]). By [Jan03, II.6.6], we know that such an x satifies $\ell(x) < \ell(ws_\delta)$.

Some of these simple G-modules may not be simple as $\mathcal{U}\mathfrak{g}$-modules if $x \bullet 0$ is not restricted. But if $\lambda = \lambda_1 + p\lambda_2$ for $\lambda_1 \in \mathbb{X}$ restricted dominant and $\lambda_2 \in \mathbb{X}$ dominant, then by Steinberg's theorem ([Jan03, II.3.17]), as $\mathcal{U}\mathfrak{g}$-modules we have $L(\lambda) \cong L(\lambda_1)^{\oplus \dim(L(\lambda_2))}$. To conclude the proof, one observes that if $v \bullet 0$ and $\nu \neq 0$ are dominant, then $\ell(t_\nu v) > \ell(v)$. □

The following proposition is "dual", in some sense, to point (ii) of Theorem 5.5.3. Recall the modules $P(w \bullet 0)$ ($w \in W^0$) defined in subsection 4.4.

Proposition 5.5.4. *Let $w \in W^0$, and $\delta \in \Phi_{\mathrm{aff}}$ such that $ws_\delta \in W^0$ and $ws_\delta \bullet 0 < w \bullet 0$. Then $R_\delta P(w \bullet 0)$ is a direct sum of $P(ws_\delta \bullet 0)$ and some $P(v \bullet 0)$ with $v \in W^0$, $\ell(v) > \ell(ws_\delta)$.*

Proof. The fact that R_δ is exact and self-adjoint implies that $R_\delta P(w \bullet 0)$ is a projective $(\mathcal{U}\mathfrak{g})_0^{\hat{0}}$-module, hence a direct sum of some $P(v \bullet 0)$ for $v \in W^0$. The multiplicity of $P(v \bullet 0)$ is the dimension of

$$\mathrm{Hom}_{\mathfrak{g}}(R_\delta P(w \bullet 0), L(v \bullet 0)) \cong \mathrm{Hom}_{\mathfrak{g}}(P(w \bullet 0), R_\delta L(v \bullet 0)).$$

Hence $\mathrm{Hom}_{\mathfrak{g}}(R_\delta P(w \bullet 0), L(v \bullet 0)) = 0$ if $vs_\delta \bullet 0 < v \bullet 0$ (in particular for $v = w$), by (4.3.2).

Assume now that $vs_\delta \bullet 0 > v \bullet 0$. Recall the definition of $Q_\delta(v)$ in (5.5.2). The exact sequences

$$Q_\delta(v) \hookrightarrow (R_\delta L(v \bullet 0))/L(v \bullet 0) \twoheadrightarrow L(v \bullet 0),$$
$$L(v \bullet 0) \hookrightarrow R_\delta L(v \bullet 0) \twoheadrightarrow (R_\delta L(v \bullet 0))/L(v \bullet 0)$$

induce an isomorphism (recall that $v \neq w$):

$$\mathrm{Hom}_{\mathfrak{g}}(P(w \bullet 0), R_\delta L(v \bullet 0)) \cong \mathrm{Hom}_{\mathfrak{g}}(P(w \bullet 0), Q_\delta(v)).$$

We know (see Theorem 5.5.3) that $Q_\delta(v)$ is semi-simple, that $L(vs_\delta \bullet 0)$ appears with multiplicity 1 in this module if $vs_\delta \bullet 0$ is restricted, and that all the other simple components have their highest weight of the form $x \bullet 0$ for $x \in W^0$ with $\ell(x) < \ell(vs_\delta)$. Hence if $\mathrm{Hom}_{\mathfrak{g}}(P(w \bullet 0), Q_\delta(v)) \neq 0$ and $v \neq ws_\delta$, then $\ell(w) < \ell(vs_\delta) = l(v) + 1$. As $\ell(ws_\delta) = \ell(w) - 1$, we obtain $\ell(v) > \ell(ws_\delta)$. For $v = ws_\delta$ we have $\mathrm{Hom}_{\mathfrak{g}}(P(w \bullet 0), Q_\delta(ws_\delta)) = \Bbbk$. $\square$

5.6 Reminder on graded algebras

We finish this section with a few facts concerning finite dimensional graded rings, to be used later.

Consider a $\mathbb{Z}$-graded $\Bbbk$-algebra A. Let $\mathrm{Mod}(A)$, resp. $\mathrm{Mod}^{\mathrm{gr}}(A)$, denote the category of A-modules, respectively of graded A-modules. Let also $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$, resp. $\mathrm{Mod}^{\mathrm{fg}}(A)$ denote the category of finitely generated graded A-modules, resp. finitely generated A-modules. As in 2.5, we denote by

$$\langle j \rangle : \mathrm{Mod}^{\mathrm{fg,gr}}(A) \to \mathrm{Mod}^{\mathrm{fg,gr}}(A)$$

the shift in the grading given by $(M\langle j \rangle)_n = M_{n-j}$. Let

$$\mathrm{For} : \mathrm{Mod}^{\mathrm{gr}}(A) \to \mathrm{Mod}(A)$$

be the forgetful functor. Following [GG82], we call *gradable* the A-modules in the essential image of this functor.

If M is in $\mathrm{Mod}(A)$, we denote by $\mathrm{rad}(M)$ the radical of M, *i.e.* the intersection of all maximal submodules in M (see *e.g.* [CR81, chapter 5]). Similarly, we denote by $\mathrm{soc}(M)$ the socle of M, *i.e.* the sum of all simple submodules of M.

In the following theorem, points (i) to (iv) are proved in [GG82, 3.2, 3.4, 3.5, 4.1]. Point (v) follows easily from the isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}(A)}(\mathrm{For}(M), \mathrm{For}(N)) \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Mod}^{\mathrm{gr}}(A)}(M, N\langle i \rangle)$$

for M and N in $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$.

Theorem 5.6.1. *Assume* $\dim_{\Bbbk}(A) < \infty$.

(i) *If* $M \in \mathrm{Mod}^{\mathrm{fg,gr}}(A)$*, then* M *is indecomposable in* $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$ *iff* $\mathrm{For}(M)$ *is indecomposable in* $\mathrm{Mod}^{\mathrm{fg}}(A)$.

(ii) *Simple and projective modules in* $\mathrm{Mod}^{\mathrm{fg}}(A)$ *are gradable.*

(iii) *If* $M \in \mathrm{Mod}^{\mathrm{fg,gr}}(A)$*, then* $\mathrm{soc}(\mathrm{For}(M))$ *and* $\mathrm{rad}(\mathrm{For}(M))$ *are homogeneous submodules.*

(iv) *If* $M, N \in \mathrm{Mod}^{\mathrm{fg,gr}}(A)$ *are indecomposable and non-zero and if* $\mathrm{For}(M) \cong \mathrm{For}(N)$*, then there exists a unique* $j \in \mathbb{Z}$ *such that* $M \cong N\langle j \rangle$ *in* $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$.

(v) *If* $M \in \mathrm{Mod}^{\mathrm{fg,gr}}(A)$*, then* M *is projective in* $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$ *iff* $\mathrm{For}(M)$ *is projective in* $\mathrm{Mod}^{\mathrm{fg}}(A)$.

The following results can be proved exactly as in the non-graded case (see also [AJS94, E.6] for a proof in a more general context):

Proposition 5.6.2. *Assume* $\dim_{\Bbbk}(A) < \infty$.

(i) *If* $M \in \mathrm{Mod}^{\mathrm{fg,gr}}(A)$*, then* M *is indecomposable in* $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$ *iff the algebra* $\mathrm{Hom}_{\mathrm{Mod}^{\mathrm{fg,gr}}(A)}(M, M)$ *is local.*

(ii) *The Krull-Schmidt theorem holds in* $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$.

These results can be used to deduce information on the structure of a graded A-module M when we know the structure of $\mathrm{For}(M)$. More precisely, assume $\dim_{\Bbbk}(A) < \infty$, and let M be in $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$. Let

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

be the decomposition of M as a sum of indecomposable submodules in the category $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$ (*i.e.* as a graded A-module). Then we have

$$\mathrm{For}(M) = \mathrm{For}(M_1) \oplus \cdots \oplus \mathrm{For}(M_n) \tag{5.6.3}$$

in $\mathrm{Mod}(A)$. Moreover, by Theorem 5.6.1(i), for all j the A-module $\mathrm{For}(M_j)$ is indecomposable. Hence (5.6.3) is the decomposition of $\mathrm{For}(M)$ as a sum of indecomposable submodules (which is unique, up to isomorphism and permutation, by the Krull-Schmidt theorem). So the M_j's are lifts of the indecomposable direct summands of $\mathrm{For}(M)$.

For later reference, let us spell out the following consequence of Theorem 5.6.1, which is implicit in [GG82] (and can also be proved directly).

Corollary 5.6.4. *Assume* $\dim_{\Bbbk}(A) < \infty$. *Let* M *be in* $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$.

(i) M *is simple in* $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$ *iff* $\mathrm{For}(M)$ *is simple in* $\mathrm{Mod}(A)$.

(ii) M *is semi-simple in the category* $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$ *iff* $\mathrm{For}(M)$ *is a semi-simple* A*-module.*

Proof. (i) It is clear that if $\mathrm{For}(M)$ is a simple A-module, then M is simple in $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$. Assume now that M is simple in $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$. Then $\mathrm{soc}(M) \subset M$ is a non-zero graded submodule by Theorem 5.6.1(iii). Hence $\mathrm{soc}(M) = M$, and M is a semi-simple A-module. As it is also indecomposable by Theorem 5.6.1(i), it is simple.

(ii) It follows from (i) that if M is semi-simple in the category $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$, then $\mathrm{For}(M)$ is a semi-simple A-module. Now assume $\mathrm{For}(M)$ is a semi-simple A-module. Choose a decomposition as a sum of indecomposable graded submodules $M = M_1 \oplus \cdots \oplus M_n$. By the remark before the corollary, $\mathrm{For}(M) = \mathrm{For}(M_1) \oplus \cdots \oplus \mathrm{For}(M_n)$ is the decomposition of M as a sum of indecomposable submodules in $\mathrm{Mod}(A)$. Hence each $\mathrm{For}(M_i)$ is simple. By (i), it follows that M_i is simple in $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$. This concludes the proof. □

6 Projective $(\mathcal{U}\mathfrak{g})_0$-modules

In this section we study in more details the right hand side of diagram $(*)$ after Proposition 3.3.14.

6.1 Geometric reflection functors

From now on, to simplify the notations we assume that G is quasi-simple, *i.e.* that R is irreducible.

We have defined the reflection functors in Definition 5.5.1. Let $\alpha \in \Phi$ be a finite simple root. Recall the definition of the morphism $\widehat{\pi}_\alpha$ in (5.4.1). By Proposition 5.4.2 the following diagram is commutative

$$\begin{tikzcd} \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[r, "L(\widehat{\pi}_\alpha)^* \circ R(\widehat{\pi}_\alpha)_*"] \arrow[d, "\wr"', "\widehat{\gamma}_0^{\mathcal{B}}"] & \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[d, "\wr"', "\widehat{\gamma}_0^{\mathcal{B}}"] \\ \mathcal{D}^b\mathrm{Mod}_0((\mathcal{U}\mathfrak{g})_0) \arrow[r, "R_\alpha"] & \mathcal{D}^b\mathrm{Mod}_0((\mathcal{U}\mathfrak{g})_0). \end{tikzcd} \tag{6.1.1}$$

For this reason, we denote by $\mathfrak{R}_\alpha$ the functor

$$L(\widehat{\pi}_\alpha)^* \circ R(\widehat{\pi}_\alpha)_* : \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \to \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}).$$

Now we want to make such a construction for the affine simple root α_0. For simplicity, sometimes we write s_0 for the corresponding simple reflection, instead of s_{α_0}. We will use the following lemma. Recall the lift $C : W'_{\mathrm{aff}} \to B'_{\mathrm{aff}}$ of the natural projection (see II.1.1).

Lemma 6.1.2. *In* B'_{aff}*, consider the lift* $C(s_0)$ *of the affine simple reflection* $s_0 \in W'_{\mathrm{aff}}$*. There exists* $\beta \in \Phi$ *and* $b_0 \in B'_{\mathrm{aff}}$ *such that*

$$C(s_0) = b_0 \cdot C(s_\beta) \cdot (b_0)^{-1}.$$

Proof. First, assume G is not of type $\mathbf{G}_2$, $\mathbf{F}_4$ or $\mathbf{E}_8$. Then $\mathbb{X}/\mathbb{Y} \neq 0$, hence there exists $\omega \in W'_{\mathrm{aff}}$ with $\ell(\omega) = 0$, but $\omega \neq 1$. Then $\omega \cdot s_0 \cdot \omega^{-1}$ is a simple reflection s_β for some $\beta \in \Phi$. As lengths add in this relation, we have also $C(s_0) = b_0 \cdot C(s_\beta) \cdot (b_0)^{-1}$ for $b_0 = C(\omega)$.

Now assume[7] G is of type $\mathbf{G}_2$, $\mathbf{F}_4$ or $\mathbf{E}_8$. Then there exists a simple root β such that the braid relation between s_0 and s_β is of length 3. Then we have $C(s_\beta)C(s_0)C(s_\beta) = C(s_0)C(s_\beta)C(s_0)$, hence

$$C(s_0) = C(s_\beta)C(s_0)C(s_\beta)C(s_0)^{-1}C(s_\beta)^{-1}.$$

Hence we can take $b_0 = C(s_\beta)C(s_0)$. □

In the rest of this chapter, we fix such a β and such a b_0.

Corollary 6.1.3. *Keep the notation of Lemma* 6.1.2. *For any* $M \in \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(0,0)}(\mathcal{U}\mathfrak{g})$, *resp.* $M \in \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_0((\mathcal{U}\mathfrak{g})_0)$, *there exists an isomorphism*[8]

$$R_{\alpha_0}(M) \cong \mathbf{I}_{b_0} \circ R_\beta \circ \mathbf{I}_{(b_0)^{-1}}(M), \quad \textit{resp.} \quad R_{\alpha_0}(M) \cong \mathbf{I}^{\mathrm{res}}_{b_0} \circ R_\beta \circ \mathbf{I}^{\mathrm{res}}_{(b_0)^{-1}}(M).$$

Proof. We only prove the first isomorphism, the second one can be proved similarly. First, Lemma 6.1.2 implies that $\mathbf{I}_{C(s_0)} \cong \mathbf{I}_{b_0} \circ \mathbf{I}_{C(s_\beta)} \circ \mathbf{I}_{(b_0)^{-1}}$. By definition of the B'_{aff}-action, for any $N \in \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(0,0)}(\mathcal{U}\mathfrak{g})$ there is an exact triangle $N \to R_\beta N \to \mathbf{I}_{C(s_\beta)}N$. Hence, for $M \in \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{(0,0)}(\mathcal{U}\mathfrak{g})$ there is an exact triangle

$$M \to \mathbf{I}_{b_0} \circ R_\beta \circ \mathbf{I}_{(b_0)^{-1}}(M) \to \mathbf{I}_{b_0} \circ \mathbf{I}_{C(s_\beta)} \circ \mathbf{I}_{(b_0)^{-1}}(M) \cong \mathbf{I}_{C(s_0)}(M).$$

On the other hand, again by definition there is an exact triangle $M \to R_{\alpha_0}M \to \mathbf{I}_{C(s_0)}M$. Identifying these two triangles we deduce the isomorphism of the corollary. □

For this reason we define the functor

$$\mathfrak{R}_{\alpha_0} : \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \to \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$$

as follows:

$$\mathfrak{R}_{\alpha_0} := \mathbf{J}^{\mathrm{dg}}_{b_0} \circ L(\widehat{\pi}_\beta)^* \circ R(\widehat{\pi}_\beta)_* \circ \mathbf{J}^{\mathrm{dg}}_{(b_0)^{-1}}$$

(see 5.2 for the notation). With this definition, by Corollary 6.1.3, the diagram analogous to (6.1.1) is commutative, at least on every object.

6.2 Dg versions of the reflection functors

Let $\alpha \in \Phi$. The dg-ringed spaces $(\mathcal{B}^{(1)}, \Lambda_{\mathcal{O}_{\mathcal{B}^{(1)}}}(\mathcal{T}^\vee_{\mathcal{B}^{(1)}}))$ and $(\mathcal{P}^{(1)}_\alpha, \Lambda_{\mathcal{O}_{\mathcal{P}^{(1)}_\alpha}}(\mathcal{T}^\vee_{\mathcal{P}^{(1)}_\alpha}))$ are naturally $\mathbb{G}_{\mathbf{m}}$-equivariant (see 1.7), and $\widehat{\pi}_\alpha$ is also $\mathbb{G}_{\mathbf{m}}$-equivariant. Easy arguments show

[7] More generally, this second proof works if G is not of type $\mathbf{C}_n$, $n \geq 2$.

[8] It is not clear from our proof whether or not these isomorphisms are functorial. However, this can be checked easily if G is not of type $\mathbf{G}_2$, $\mathbf{F}_4$ or $\mathbf{E}_8$.

that the functors $R(\widehat{\pi}_{\alpha,\mathbb{G}_\mathbf{m}})_*$ and $L(\widehat{\pi}_{\alpha,\mathbb{G}_\mathbf{m}})^*$ restrict to functors between the categories $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$ and $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}}_\alpha \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}_\alpha} \mathcal{P}_\alpha)^{(1)})$, with usual compatibility conditions. Equivalently, these dg-schemes and morphism can be realized using the first equivalence of Lemma 5.2.3 (and an analogue for $\mathcal{P}_\alpha$). We define

$$\mathfrak{R}_\alpha^{\mathrm{gr}} := L(\widehat{\pi}_{\alpha,\mathbb{G}_\mathbf{m}})^* \circ R(\widehat{\pi}_{\alpha,\mathbb{G}_\mathbf{m}})_*.$$

This is an endofunctor of $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$.

For the affine simple root α_0 we define similarly, with the notation of Lemma 6.1.2,

$$\mathfrak{R}_{\alpha_0}^{\mathrm{gr}} := \mathbf{J}_{b_0}^{\mathrm{dg,gr}} \circ L(\widehat{\pi}_{\beta,\mathbb{G}_\mathbf{m}})^* \circ R(\widehat{\pi}_{\beta,\mathbb{G}_\mathbf{m}})_* \circ \mathbf{J}_{(b_0)^{-1}}^{\mathrm{dg,gr}}. \tag{6.2.1}$$

With these definitions, for any $\delta \in \Phi_{\mathrm{aff}}$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[r, "\mathfrak{R}_\delta^{\mathrm{gr}}"] \arrow[d, "\mathrm{For}"'] & \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[d, "\mathrm{For}"] \\
\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[r, "\mathfrak{R}_\delta"] & \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}).
\end{tikzcd} \tag{6.2.2}$$

To conclude this subsection, for later use we study the relation between the functor $\mathfrak{R}_\alpha$ for $\alpha \in \Phi$ and the action of the braid group. Consider the following diagram of $\mathbb{G}_\mathbf{m}$-equivariant dg-schemes:

$$\begin{tikzcd}
& \big((\widetilde{\mathfrak{g}} \times_{\widetilde{\mathfrak{g}}_\alpha} \widetilde{\mathfrak{g}}) \overset{R}{\cap}_{\mathfrak{g}^* \times (\mathcal{B} \times \mathcal{B})} (\mathcal{B} \times \mathcal{B})\big)^{(1)} \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\
\big(\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B}\big)^{(1)} \arrow[dr, "\widehat{\pi}_\alpha"'] & & \big(\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B}\big)^{(1)}. \arrow[dl, "\widehat{\pi}_\alpha"] \\
& \big(\widetilde{\mathfrak{g}}_\alpha \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}_\alpha} \mathcal{P}_\alpha\big)^{(1)} &
\end{tikzcd}$$

Here we consider the realization of the dg-schemes given by the first equivalence of Lemma 5.2.3 (and analogues for the other dg-schemes). We want to construct an isomorphism of endofunctors of $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$:

$$L(\widehat{\pi}_{\alpha,\mathbb{G}_\mathbf{m}})^* \circ R(\widehat{\pi}_{\alpha,\mathbb{G}_\mathbf{m}})_* \cong R(q_{2,\mathbb{G}_\mathbf{m}})_* \circ L(q_{1,\mathbb{G}_\mathbf{m}})^*. \tag{6.2.3}$$

There is a natural adjunction morphism $\mathrm{Id} \to R(q_{1,\mathbb{G}_\mathbf{m}})_* \circ L(q_{1,\mathbb{G}_\mathbf{m}})^*$. Applying the functor $R(\widehat{\pi}_{\alpha,\mathbb{G}_\mathbf{m}})_*$ to this morphism, and using the equality $\widehat{\pi}_\alpha \circ q_1 = \widehat{\pi}_\alpha \circ q_2$, one obtains a morphism $R(\widehat{\pi}_{\alpha,\mathbb{G}_\mathbf{m}})_* \to R(\widehat{\pi}_{\alpha,\mathbb{G}_\mathbf{m}})_* \circ R(q_{2,\mathbb{G}_\mathbf{m}})_* \circ L(q_{1,\mathbb{G}_\mathbf{m}})^*$. Now, applying again adjunction, one obtains the desired morphism

$$L(\widehat{\pi}_{\alpha,\mathbb{G}_\mathbf{m}})^* \circ R(\widehat{\pi}_{\alpha,\mathbb{G}_\mathbf{m}})_* \to R(q_{2,\mathbb{G}_\mathbf{m}})_* \circ L(q_{1,\mathbb{G}_\mathbf{m}})^*.$$

Under the functor $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \xrightarrow{R(p_{\mathbb{G}_{\mathbf{m}}})_*} \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathfrak{g}}^{(1)}) \xrightarrow{\mathrm{For}} \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})$, this morphism corresponds to the isomorphism considered in Proposition II.6.1.2. Hence it is also an isomorphism (recall that $R(p_{\mathbb{G}_{\mathbf{m}}})_*$ is a forgetful functor).

Recall the shift functor $\langle 1 \rangle$ defined in subsection 2.5 (see also 5.2). The following lemma follows immediately from isomorphism (6.2.3) and the exact sequence of $\mathbb{G}_{\mathbf{m}}$-equivariant sheaves (5.3.5).

Lemma 6.2.4. *There exists a distinguished triangle of functors*

$$\mathrm{Id}\langle 1 \rangle \to \mathfrak{R}_\alpha^{\mathrm{gr}}\langle -1 \rangle \to \mathbf{J}_{T_\alpha}^{\mathrm{dg,gr}}.$$

6.3 Gradings

As in subsection 3.3, for simplicity we denote the variety $\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$ by X in this subsection. Recall the algebra $\widetilde{U} := \mathcal{U}\mathfrak{g} \otimes_{\mathfrak{Z}_{\mathrm{HC}}} S(\mathfrak{h})$, also considered in 3.3. By [BMR08, 3.4.1] we have

$$R^i\Gamma(X, \widetilde{\mathcal{D}}) \cong \begin{cases} \widetilde{U} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widetilde{U}_{\hat{0}}^{\hat{0}}$ denote the completion of $\widetilde{U}$ with respect to the maximal ideal of its center $\mathfrak{Z} \otimes_{\mathfrak{Z}_{\mathrm{HC}}} \mathrm{S}(\mathfrak{h})$ generated by $\mathfrak{h}$ and $\mathfrak{g}^{(1)}$. Let also $(\mathcal{U}\mathfrak{g})_{\hat{0}}^{\hat{0}}$ denote the completion of $\mathcal{U}\mathfrak{g}$ with respect to the maximal ideal of $\mathfrak{Z}$ corresponding to the character $(0, 0)$. The projection $\mathfrak{h}^* \to \mathfrak{h}^*/(W, \bullet)$ induces an isomorphism $\widetilde{U}_{\hat{0}}^{\hat{0}} \cong (\mathcal{U}\mathfrak{g})_{\hat{0}}^{\hat{0}}$. Recall that we have defined the algebra $(\mathcal{U}\mathfrak{g})_0^{\hat{0}}$ in subsection 4.4.

As in subsection 3.3 we let $\widehat{\mathcal{B}^{(1)}}$ denote the formal neighborhood of $\mathcal{B}^{(1)} \times \{0\}$ in $\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$. Applying [Gro61b, 4.1.5] to the proper morphism

$$\widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^* \to \mathfrak{g}^{*(1)} \times_{\mathfrak{h}^{*(1)}/W} \mathfrak{h}^*,$$

and using the fact that $\mathfrak{g}^{*(1)} \times_{\mathfrak{h}^{*(1)}/W} \mathfrak{h}^*$ is affine, we obtain isomorphisms

$$R^i\Gamma(\widehat{\mathcal{B}^{(1)}}, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}}) \cong \begin{cases} (\mathcal{U}\mathfrak{g})_{\hat{0}}^{\hat{0}} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3.1}$$

Recall also the isomorphism of sheaves of algebras on $\widehat{\mathcal{B}^{(1)}}$ (see subsection I.1.2)

$$\widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \cong \mathcal{E}nd_{\mathcal{O}_{\widehat{\mathcal{B}^{(1)}}}}(\mathcal{M}^0). \tag{6.3.2}$$

Let $\mathfrak{Z}_{\mathrm{Fr}}^+$ denote the maximal ideal of $\mathfrak{Z}_{\mathrm{Fr}}$ associated to the character 0. There is a surjection

$$(\mathcal{U}\mathfrak{g})_{\hat{0}}^{\hat{0}} \twoheadrightarrow (\mathcal{U}\mathfrak{g})_{\hat{0}}^{\hat{0}}/\langle \mathfrak{Z}_{\mathrm{Fr}}^+ \rangle \cong (\mathcal{U}\mathfrak{g})_0^{\hat{0}}.$$

Hence the algebra $(\mathcal{U}\mathfrak{g})_0^{\hat{0}}$ is a quotient of $(\mathcal{U}\mathfrak{g})_{\hat{0}}^{\hat{0}} \cong \Gamma(\widehat{\mathcal{B}^{(1)}}, \mathcal{E}nd_{\mathcal{O}_{\widehat{\mathcal{B}^{(1)}}}}(\mathcal{M}^0))$.

Let Y be a noetherian scheme and $Z \subset Y$ be a closed subscheme, with corresponding ideal $\mathcal{I}_Z \subset \mathcal{O}_Y$. Let $\widehat{Z}$ be the formal neighborhood of Z in Y (a formal scheme). Assume $\widehat{Z}$ is endowed with a $\mathbb{G}_\mathbf{m}$-action. If $\mathcal{F}$ is a coherent sheaf on $\widehat{Z}$, a structure of $\mathbb{G}_\mathbf{m}$-equivariant coherent sheaf on $\mathcal{F}$ is the data, for any n, of a structure of $\mathbb{G}_\mathbf{m}$-equivariant coherent sheaf on $\mathcal{F}/(\mathcal{I}_Z^n \cdot \mathcal{F})$ (as a coherent sheaf on the n-th infinitesimal neighborhood of Z in Y), all these structures being compatible. Let us recall the following result, due to V. Vologodsky (see the second appendix in the preprint version of [BFG06]):

Lemma 6.3.3. *Let $f : Y \to Z$ be a proper morphism of $\Bbbk$-schemes. Let z be a point in Z, and $Y_{\hat{z}}$ be the formal neighborhood of $f^{-1}(z)$ in Y. Let $\mathcal{E}$ be a vector bundle on $Y_{\hat{z}}$, such that* $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) = 0$*. If $Y_{\hat{z}}$ is endowed with a $\mathbb{G}_\mathbf{m}$-action, then this action can be lifted to $\mathcal{E}$,* i.e. *there exists a $\mathbb{G}_\mathbf{m}$-equivariant structure on $\mathcal{E}$.*

Now we consider $\widehat{\mathcal{B}^{(1)}}$ as the formal neighborhood of the zero-section in $\widetilde{\mathfrak{g}}^{(1)}$. We have defined a $\mathbb{G}_\mathbf{m}$-action on $\widetilde{\mathfrak{g}}^{(1)}$ in (5.2.1). This action stabilizes the zero-section, hence induces an action on $\widehat{\mathcal{B}^{(1)}}$. We can apply Lemma 6.3.3 to the splitting bundle $\mathcal{M}^0$, the vanishing hypothesis following from (6.3.1) and (6.3.2). Hence we obtain a $\mathbb{G}_\mathbf{m}$-equivariant structure on $\mathcal{M}^0$, and a structure of a $\mathbb{G}_\mathbf{m}$-equivariant sheaf of algebras on $\widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}}$.

Applying $\Gamma(\widehat{\mathcal{B}^{(1)}}, -)$, we obtain a $\mathbb{G}_\mathbf{m}$-equivariant algebra structure on $(\mathcal{U}\mathfrak{g})^{\hat{0}}_{\hat{0}}$, which is compatible with the $\mathbb{G}_\mathbf{m}$-structure on $\mathfrak{g}^{*(1)}$ induced by the action on $\widetilde{\mathfrak{g}}^{(1)}$. Taking the quotient (by a homogeneous ideal), we obtain a grading on the algebra $(\mathcal{U}\mathfrak{g})^{\hat{0}}_0$. Let $\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})_0)$ denote the category of finitely generated graded modules over this graded algebra.

The following theorem is a "graded version" of Theorem 3.3.3:

Theorem 6.3.4. *There exists a fully faithful triangulated functor*

$$\widetilde{\gamma}_0^{\mathcal{B}} : \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \ \to \ \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})_0),$$

commuting with the internal shifts $\langle 1 \rangle$, and such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[r, "\widetilde{\gamma}_0^{\mathcal{B}}"] \arrow[d, "\mathrm{For}"'] & \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})_0) \arrow[d, "\mathrm{For}"] \\
\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[r, "\widetilde{\gamma}_0^{\mathcal{B}}"] & \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0).
\end{tikzcd}$$

This theorem would be easy to prove if we had a $\mathbb{G}_\mathbf{m}$-equivariant structure on the whole of $\widetilde{\mathcal{D}}$ and $\mathcal{U}\mathfrak{g}$ (the proof of Theorem 3.3.3 would generalize in a straightforward manner, and we would even obtain an equivalence of categories). Unfortunately we only have such a structure on some completions of these algebras, and this subtlety complicates the proof. As it is long and as the details are not needed, we postpone the proof of Theorem 6.3.4 to the end of this section (see 6.6 and 6.7).

Remark 6.3.5. Arguing as in the proof of Proposition 7.2.3 below, one can prove that the functor $\widetilde{\gamma}_0^{\mathcal{B}}$ is essentially surjective, hence an equivalence (see Remark 7.2.4 for the "dual" statement).

6.4 Complexes representing a projective module

The abelian category $\mathrm{Mod}^{\mathrm{fg}}_{(0,0)}(\mathcal{U}\mathfrak{g})$ does not have any projective object (because of the assumption that the center acts with a *generalized* character on these modules). Nevertheless, in the category $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})$ one can define the notion of a complex of sheaves "representing a projective module". For $\mathcal{F}, \mathcal{G} \in \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})$, we write simply $\mathrm{Hom}_{\widetilde{\mathfrak{g}}^{(1)}}(\mathcal{F}, \mathcal{G})$ for $\mathrm{Hom}_{\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})}(\mathcal{F}, \mathcal{G})$. The following definition was already considered (in a special case) in I.2.3.

Definition 6.4.1. Let $\lambda \in \mathbb{X}$ be regular. An object $\mathcal{M}$ of $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})$ is said to *represent a projective module under* $\gamma^{\mathcal{B}}_{\lambda}$ if

$$\mathrm{Hom}_{\widetilde{\mathfrak{g}}^{(1)}}(\mathcal{M}, (\gamma^{\mathcal{B}}_{\lambda})^{-1} N[i]) = 0$$

for any $N \in \mathrm{Mod}^{\mathrm{fg}}_{(\lambda,0)}(\mathcal{U}\mathfrak{g})$ and $i \neq 0$.

Let $\mu \in \mathbb{X}$ be a restricted dominant weight in the orbit of λ under W'_{aff}. An object $\mathcal{M}$ of $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})$ is said to *represent the projective cover of $L(\mu)$ under $\gamma^{\mathcal{B}}_{\lambda}$* if for any $\nu \in W'_{\mathrm{aff}} \bullet \lambda$ restricted and dominant and $i \in \mathbb{Z}$,

$$\mathrm{Hom}_{\widetilde{\mathfrak{g}}^{(1)}}(\mathcal{M}, (\gamma^{\mathcal{B}}_{\lambda})^{-1} L(\nu)[i]) = \begin{cases} \Bbbk & \text{if } \nu = \mu \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Recall from subsection 4.1 the element $\tau_0 = t_\rho \cdot w_0 \in W^0 \subset W'_{\mathrm{aff}}$.

Lemma 6.4.2. *Let $\lambda \in C_0$, and $v \in W^0$. Then $T^{-\rho}_{\lambda} L(v \bullet \lambda) \neq 0$ iff $v = \tau_0$. Moreover, $T^{-\rho}_{\lambda} L(\tau_0 \bullet \lambda) = L(\tau_0 \bullet (-\rho)) = L((p-1)\rho)$.*

Proof. Using the rule given by (4.3.2) to compute $T^{-\rho}_{\lambda} L(v \bullet \lambda)$, we only have to prove that $v \bullet (-\rho)$ is in the upper closure of $v \bullet C_0$ if and only if $v = \tau_0$. Write $v = t_\nu \cdot w$ with $\nu \in \mathbb{X}$, $w \in W$. Then one easily checks that $v \bullet (-\rho)$ is in the upper closure of $v \bullet C_0$ if and only if $w = w_0$. The result follows since, under the assumption that $v \bullet \lambda$ is dominant restricted, ν is uniquely determined by w (see equation (4.1.3)). □

Proposition 6.4.3. *Let $\lambda \in C_0$, and $w \in W$. The object $\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}$ represents the projective cover of $L(\tau_0 \bullet \lambda)$ under $\gamma^{\mathcal{B}}_{w \bullet \lambda}$.*

Proof. Consider the functor $T^{-\rho}_{\lambda} = T^{w \bullet (-\rho)}_{w \bullet \lambda} = T^{-\rho}_{w \bullet \lambda}$. By Proposition I.1.3.1 applied to the weights $w \bullet \lambda$ and $-\rho$, with $\mathcal{P} = G/G = \{\mathrm{pt}\}$, we have

$$T^{-\rho}_{w \bullet \lambda} \circ \gamma^{\mathcal{B}}_{w \bullet \lambda} \cong \gamma^{\{\mathrm{pt}\}}_{-\rho} \circ R\Gamma(\widetilde{\mathfrak{g}}^{(1)}, -). \tag{6.4.4}$$

Moreover, $\mathrm{Hom}_{\widetilde{\mathfrak{g}}^{(1)}}(\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}, -) \cong H^0(R\Gamma(\widetilde{\mathfrak{g}}^{(1)}, -))$. Now the result follows from (6.4.4) and Lemma 6.4.2, using the fact that $\gamma^{\{\mathrm{pt}\}}_{-\rho}(\Bbbk) = L((p-1)\rho)$. (The latter fact can be proved either by looking at the definition of the splitting bundles, see [BMR06, 1.3.5], or by remarking that $L((p-1)\rho)$ is the only simple module in $\mathrm{Mod}^{\mathrm{fg}}_{(-\rho,0)}(\mathcal{U}\mathfrak{g})$.) □

Corollary 6.4.5. *Let $\lambda \in (W'_{\mathrm{aff}} \bullet 0) \cap C_0$. Write $\lambda = \omega \bullet 0$ for $\omega = w \cdot t_\mu \in W'_{\mathrm{aff}}$ ($\mu \in \mathbb{X}$, $w \in W$). Then $\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}(\mu)$ represents the projective cover of $L(\tau_0 \bullet \lambda)$ under $\gamma_0^{\mathcal{B}}$.*

Proof. By hypothesis, $\lambda = \omega \bullet 0 = w \bullet (p\mu)$. Hence $w^{-1} \bullet \lambda = p\mu$. By Proposition 6.4.3, $\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}$ represents the projective cover of $L(\tau_0 \bullet \lambda)$ under $\gamma^{\mathcal{B}}_{w^{-1}\bullet\lambda} = \gamma^{\mathcal{B}}_{p\mu}$. But for $\mathcal{F} \in \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)})$ we have $\gamma^{\mathcal{B}}_{p\mu}(\mathcal{F}) = \gamma^{\mathcal{B}}_0(\mathcal{F} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}} \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}(\mu))$ (see I.1.2). The result follows. □

Recall that we have defined the objects $P(w \bullet 0)$, $\mathcal{P}_w$ in subsection 4.4. Consider the natural morphism of dg-schemes $p : (\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)} \to \widetilde{\mathfrak{g}}^{(1)}$. By adjunction, it is clear that if $\mathcal{M} \in \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathfrak{g}}^{(1)})$ represents a projective module under $\gamma_0^{\mathcal{B}}$, then $\widehat{\gamma}_0^{\mathcal{B}}(Lp^*\mathcal{M})$ is a projective $(\mathcal{U}\mathfrak{g})_0^{\hat{0}}$-module. In particular, with the notation of Corollary 6.4.5, we have

$$\mathcal{O}_{(\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}}(\mu) \cong \mathcal{P}_{\tau_0\omega}. \tag{6.4.6}$$

6.5 Graded projective $(\mathcal{U}\mathfrak{g})_0$-modules

Recall the results of subsection 5.6. Using Theorem 5.6.1 (ii) and (iv), the projective modules $P(w \bullet 0)$ can be lifted to graded modules (uniquely, up to a shift). In this subsection we fix an arbitrary choice of a lift for each $P(w \bullet 0)$, and denote it by $P^{\mathrm{gr}}(w \bullet 0)$. Recall the fully faithful functor

$$\widetilde{\gamma}_0^{\mathcal{B}} : \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \to \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})_0)$$

of Theorem 6.3.4.

Proposition 6.5.1. *For all $w \in W^0$, $P^{\mathrm{gr}}(w \bullet 0)$ is in the essential image of the functor $\widetilde{\gamma}_0^{\mathcal{B}}$.*

Proof. We prove the result by descending induction on $\ell(w)$. By Proposition 4.1.2, the elements $w \in W^0$ such that $\ell(w)$ is maximal are of the form $w = \tau_0\omega$, for $\omega \in W'_{\mathrm{aff}}$ such that $\ell(\omega) = 0$. In this case, by (6.4.6) we have $\mathcal{P}_{\tau_0\omega} \cong \mathcal{O}_{(\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}}(\mu)$ (with the notation of Corollary 6.4.5). It is clear that $\mathcal{O}_{(\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}}(\mu)$ can be considered as an object of $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$. By Theorem 5.6.1(iv) and the commutative diagram in Theorem 6.3.4, the image of this object under $\widetilde{\gamma}_0^{\mathcal{B}}$ is isomorphic to $P^{\mathrm{gr}}(\tau_0\omega \bullet 0)$, up to a shift. As $\widetilde{\gamma}_0^{\mathcal{B}}$ commutes with the internal shift, this proves the result when $\ell(w) = \ell(\tau_0)$.

Now let n be a non-negative integer such that $n < \ell(\tau_0)$, and assume the result is true for all $v \in W^0$ such that $\ell(v) > n$. Let $w \in W^0$ be such that $\ell(w) = n$. Let $\delta \in \Phi_{\mathrm{aff}}$ be such that $ws_\delta \in W^0$ and $ws_\delta \bullet 0 > w \bullet 0$, *i.e.* $\ell(ws_\delta) > \ell(w)$. By induction, there exists $\mathcal{P}^{\mathrm{gr}}$ in $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$ such that $\widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{P}^{\mathrm{gr}}) \cong P^{\mathrm{gr}}(ws_\delta \bullet 0)$. Then consider

$$\widetilde{\gamma}_0^{\mathcal{B}}(\mathfrak{R}_\delta^{\mathrm{gr}}\mathcal{P}^{\mathrm{gr}}).$$

By construction, using diagrams (6.1.1) and (6.2.2), the image of this object under the forgetful functor

$$\mathrm{For} : \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})_0) \to \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$$

is $R_\delta P(ws_\delta \bullet 0)$. In particular $\widetilde{\gamma}_0^{\mathcal{B}}(\mathfrak{R}_\delta^{\mathrm{gr}}\mathcal{P}^{\mathrm{gr}})$ is concentrated in degree 0, *i.e.* is a graded $(\mathcal{U}\mathfrak{g})_0^{\hat{0}}$-module. By Proposition 5.5.4, $R_\delta P(ws_\delta \bullet 0)$ is a direct sum of $P(w \bullet 0)$ and some $P(v \bullet 0)$ with $v \in W^0$ such that $\ell(v) > \ell(w)$. Hence, using the remark before Corollary 5.6.4, $\widetilde{\gamma}_0^{\mathcal{B}}(\mathfrak{R}_\delta^{\mathrm{gr}}\mathcal{P}^{\mathrm{gr}}) \cong P^{\mathrm{gr}}(w \bullet 0)\langle i\rangle \oplus Q^{\mathrm{gr}}$ for some $i \in \mathbb{Z}$, where Q^{gr} is a direct sum of graded modules of the form $P^{\mathrm{gr}}(v \bullet 0)\langle j\rangle$ with $j \in \mathbb{Z}$ and $v \in W^0$ such that $\ell(v) > \ell(w)$. By induction hypothesis, there exists an object $\mathcal{Q}^{\mathrm{gr}}$ in $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$ such that $Q^{\mathrm{gr}} \cong \widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{Q}^{\mathrm{gr}})$. Then we have

$$\widetilde{\gamma}_0^{\mathcal{B}}(\mathfrak{R}_\delta^{\mathrm{gr}}\mathcal{P}^{\mathrm{gr}}) \;\cong\; \widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{Q}^{\mathrm{gr}}) \oplus P^{\mathrm{gr}}(w \bullet 0)\langle i\rangle.$$

As $\widetilde{\gamma}_0^{\mathcal{B}}$ is fully faithful, the natural injection $\widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{Q}^{\mathrm{gr}}) \hookrightarrow \widetilde{\gamma}_0^{\mathcal{B}}(\mathfrak{R}_\delta^{\mathrm{gr}}\mathcal{P}^{\mathrm{gr}})$ comes from a morphism $\mathcal{Q}^{\mathrm{gr}} \to \mathfrak{R}_\delta^{\mathrm{gr}}\mathcal{P}^{\mathrm{gr}}$ in $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$. Let $\mathcal{X}^{\mathrm{gr}}$ be the cone of this morphism. Then, by usual properties of triangulated categories, there exists an isomorphism

$$\widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{X}^{\mathrm{gr}}\langle -i\rangle) \;\cong\; P^{\mathrm{gr}}(w \bullet 0).$$

This concludes the proof of the induction step, and of the proposition. □

6.6 Some generalities on $\mathbb{G}_{\mathbf{m}}$-equivariant quasi-coherent dg-modules

In the next two subsections we prove Theorem 6.3.4. We begin with some general results on $\mathbb{G}_{\mathbf{m}}$-equivariant quasi-coherent dg-modules.

Let us consider a noetherian scheme A, and a non-positively graded, $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebra $\mathcal{A}$ on A (as in 1.7). Assume also that $\mathcal{A}$ is locally finitely generated as an $\mathcal{O}_A$-algebra. Let $\mathcal{D}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(A, \mathcal{A})$, respectively $\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_{\mathbf{m}}}(A, \mathcal{A})$, be the full subcategory of $\mathcal{D}_{\mathbb{G}_{\mathbf{m}}}(A, \mathcal{A})$ whose objects have their cohomology quasi-coherent over $\mathcal{O}_A$, resp. quasi-coherent over $\mathcal{O}_A$ and locally finitely generated over $H(\mathcal{A})$. Let also $\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(A, \mathcal{A})$ be the category of $\mathbb{G}_{\mathbf{m}}$-equivariant dg-modules which are quasi-coherent over $\mathcal{O}_A$, and let $\mathcal{D}\big(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(A, \mathcal{A})\big)$ be the corresponding derived category (the localization of the homotopy category of $\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(A, \mathcal{A})$). Let $\mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(A, \mathcal{A})\big)$ be the full subcategory of $\mathcal{D}\big(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(A, \mathcal{A})\big)$ whose objects have their cohomology locally finitely generated over $H(\mathcal{A})$.

A proof entirely similar to that of Lemma 3.2.2 (here we do not consider any condition on the support) (see also Lemma 1.7.1) shows that if $\mathcal{F}$ is an object of $\mathcal{D}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(A, \mathcal{A})$ whose cohomology is bounded, there exists a $\mathbb{G}_{\mathbf{m}}$-equivariant K-injective $\mathcal{A}$-dg-module $\mathcal{I}$ and a quasi-isomorphism $\mathcal{F} \to \mathcal{I}$, where $\mathcal{I}$ is quasi-coherent over $\mathcal{O}_A$. We deduce the following:

Lemma 6.6.1. *Assume $\mathcal{A}$ is bounded for the cohomological grading. There exists an equivalence of categories*

$$\mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(A, \mathcal{A})\big) \;\cong\; \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_{\mathbf{m}}}(A, \mathcal{A}).$$

Now, let Y be a noetherian scheme equipped with a (possibly non trivial) $\mathbb{G}_{\mathbf{m}}$-action. In the rest of this subsection we consider two different situations, denoted (a) and (b).

Situation (a) is the following. Let $\mathcal{Y}$ be a dg-algebra on Y (non-positively graded). We have not defined $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebras and dg-modules in this case. But assume that $\mathcal{Y}$ is *coherent* as an $\mathcal{O}_Y$-module, and that each $\mathcal{Y}^p$ is equipped with a $\mathbb{G}_\mathbf{m}$-equivariant structure (as a coherent $\mathcal{O}_Y$-module), such that the multiplication and the differential are $\mathbb{G}_\mathbf{m}$-equivariant.

Then we can consider the notion of an *$\mathcal{O}_Y$-quasi-coherent*, *$\mathbb{G}_\mathbf{m}$-equivariant* dg-module over $\mathcal{Y}$. We denote by $\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_\mathbf{m}}(Y,\mathcal{Y})$ the corresponding category, and by $\mathcal{C}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(Y,\mathcal{Y})$ the full subcategory of dg-modules locally finitely generated over $\mathcal{Y}$. We denote the corresponding derived categories by $\mathcal{D}\big(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_\mathbf{m}}(Y,\mathcal{Y})\big)$ and $\mathcal{D}\big(\mathcal{C}^{\mathrm{qc,fg}}_{\mathbb{G}_\mathbf{m}}(Y,\mathcal{Y})\big)$. We also denote by $\mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_\mathbf{m}}(Y,\mathcal{Y})\big)$ the full subcategory of $\mathcal{D}\big(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_\mathbf{m}}(Y,\mathcal{Y})\big)$ whose objects have locally finitely generated cohomology.

Consider a closed $\mathbb{G}_\mathbf{m}$-subscheme $Z\subset Y$. Denote by $\mathcal{D}^{\mathrm{fg}}_Z\big(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_\mathbf{m}}(Y,\mathcal{Y})\big)$ the full subcategory of $\mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_\mathbf{m}}(Y,\mathcal{Y})\big)$ whose objects have their cohomology supported on Z. We also consider the category $\mathcal{C}^{\mathrm{qc},\mathbb{G}_\mathbf{m}}_Z(Y,\mathcal{Y})$ of $\mathbb{G}_\mathbf{m}$-equivariant, quasi-coherent $\mathcal{Y}$-dg-modules supported on Z, its subcategory $\mathcal{C}^{\mathrm{qc,fg},\mathbb{G}_\mathbf{m}}_Z(Y,\mathcal{Y})$, the derived categories $\mathcal{D}\big(\mathcal{C}^{\mathrm{qc},\mathbb{G}_\mathbf{m}}_Z(Y,\mathcal{Y})\big)$, $\mathcal{D}\big(\mathcal{C}^{\mathrm{qc,fg},\mathbb{G}_\mathbf{m}}_Z(Y,\mathcal{Y})\big)$, and the full subcategory $\mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc},\mathbb{G}_\mathbf{m}}_Z(Y,\mathcal{Y})\big)$ of $\mathcal{D}\big(\mathcal{C}^{\mathrm{qc},\mathbb{G}_\mathbf{m}}_Z(Y,\mathcal{Y})\big)$ of objects having locally finitely generated cohomology.

Now we consider situation (b). As above, let $Z\subset Y$ be a closed $\mathbb{G}_\mathbf{m}$-subscheme. Let $\widehat{\mathcal{Y}}$ be a coherent sheaf of dg-algebras on the formal neighborhood $\widehat{Z}$ of Z in Y, endowed with a $\mathbb{G}_\mathbf{m}$-equivariant structure. Hence, if $\mathcal{I}_Z$ is the defining ideal of Z in Y, we have a $\mathbb{G}_\mathbf{m}$-equivariant structure on the quotient $\widehat{\mathcal{Y}}/(\mathcal{I}_Z^n\cdot\widehat{\mathcal{Y}})$ for any $n>0$, and all these structures are compatible. Then we can define the abelian category $\mathcal{C}^{\mathrm{qc},\mathbb{G}_\mathbf{m}}_Z(Y,\widehat{\mathcal{Y}})$ whose objects are quasi-coherent, $\mathbb{G}_\mathbf{m}$-equivariant $\mathcal{O}_Y$-dg-modules supported on Z, endowed with a compatible action of $\widehat{\mathcal{Y}}$ (by definition such an object is a direct limit of dg-modules over some quotients $\widehat{\mathcal{Y}}/(\mathcal{I}_Z^n\cdot\widehat{\mathcal{Y}})$ for $n\gg 0$). We use the same notation as above for the categories of locally finitely generated dg-modules, and for the derived categories.

Observe that situation (b) is a particular case of situation (a) (taking $\widehat{\mathcal{Y}}$ to be a the restriction of $\mathcal{Y}$ to $\widehat{Z}$). The notations are compatible.

Recall the construction of resolutions by injective $\mathbb{G}_\mathbf{m}$-equivariant quasi-coherent sheaves on Y (see *e.g.* [Bez00]): if $\mathcal{F}$ is an injective object of $\mathrm{QCoh}(Y)$, then $\mathrm{Av}(\mathcal{F}):=a_*p_Y^*\mathcal{F}$ is injective in $\mathrm{QCoh}^{\mathbb{G}_\mathbf{m}}(Y)$, where a and $p_Y:\mathbb{G}_\mathbf{m}\times Y\to Y$ are the action and the projection, respectively. It follows from this construction, using the non-equivariant case (see [BMR06, 3.1.7]), that any $\mathbb{G}_\mathbf{m}$-equivariant quasi-coherent sheaf on Y which is supported on Z can be embedded into an injective $\mathbb{G}_\mathbf{m}$-equivariant quasi-coherent sheaf supported on Z.

Using these remarks, arguments similar to those of the proof of Proposition 3.2.4 (here the situation is easier, because we only consider *quasi-coherent* dg-modules) give:

Lemma 6.6.2. (i) *Assume we are in situation* (b). *Then there exists an equivalence of categories*

$$\mathcal{D}\big(\mathcal{C}^{\mathrm{qc,fg},\mathbb{G}_\mathbf{m}}_Z(Y,\widehat{\mathcal{Y}})\big)\ \cong\ \mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc},\mathbb{G}_\mathbf{m}}_Z(Y,\widehat{\mathcal{Y}})\big).$$

(ii) *Assume we are in situation* (a). *Then there exists an equivalence of categories*

$$\mathcal{D}\big(\mathcal{C}^{\mathrm{qc,fg},\mathbb{G}_\mathbf{m}}_Z(Y,\mathcal{Y})\big)\ \cong\ \mathcal{D}^{\mathrm{fg}}_Z\big(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_\mathbf{m}}(Y,\mathcal{Y})\big).$$

As in subsection 2.5, we denote by $\langle 1\rangle$ the shift in the internal grading (*i.e.* the tensor product with the 1-dimensional $\mathbb{G}_{\mathbf{m}}$-module given by $\mathrm{Id}_{\mathbb{G}_{\mathbf{m}}}$).

Lemma 6.6.3. (i) *Assume we are in situation (a). Let $\mathcal{F},\mathcal{G}$ be objects of $\mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(Y,\mathcal{Y})\big)$. There exists an isomorphism*

$$\bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^{\mathrm{fg}}\left(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(Y,\mathcal{Y})\right)}(\mathcal{F},\mathcal{G}\langle m\rangle) \;\cong\; \mathrm{Hom}_{\mathcal{D}^{\mathrm{qc,fg}}(Y,\mathcal{Y})}(\mathrm{For}\,\mathcal{F},\mathrm{For}\,\mathcal{G}),$$

where For *is the forgetful functor.*

(ii) *Assume we are in situation (b). Let $\mathcal{F},\mathcal{G}$ be objects of $\mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{Z}(Y,\widehat{\mathcal{Y}})\big)$. There exists an isomorphism*

$$\bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^{\mathrm{fg}}\left(\mathcal{C}^{\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{Z}(Y,\widehat{\mathcal{Y}})\right)}(\mathcal{F},\mathcal{G}\langle m\rangle) \;\cong\; \mathrm{Hom}_{\mathcal{D}^{\mathrm{fg}}\left(\mathcal{C}^{\mathrm{qc}}_{Z}(Y,\widehat{\mathcal{Y}})\right)}(\mathrm{For}\,\mathcal{F},\mathrm{For}\,\mathcal{G}),$$

where For *is the forgetful functor, and the category on the right hand side has the obvious definition.*

Proof. (i) Using an open affine covering, we can assume Y is affine, hence consider categories of modules over a dg-algebra rather than sheaves of dg-modules over sheaves of dg-algebras (see Proposition 3.2.4 for the category $\mathcal{D}^{\mathrm{qc,fg}}(Y,\mathcal{Y})$). By Lemma 6.6.2(ii), we can assume $\mathcal{G}$ is finitely generated. Using a truncation functor, we can assume $\mathcal{F}$ is bounded above. Using the remarks before Lemma 3.3.6 and the construction of K-projective resolutions as in [BL94, 10.12], we can even assume that $\mathcal{F}^p$ is finitely generated over $\mathcal{Y}^0$ for any p, that for all $m\in\mathbb{Z}$ we have

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{fg}}\left(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(Y,\mathcal{Y})\right)}(\mathcal{F},\mathcal{G}\langle m\rangle) \;\cong\; \mathrm{Hom}_{\mathcal{H}^{\mathrm{fg}}\left(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(Y,\mathcal{Y})\right)}(\mathcal{F},\mathcal{G}\langle m\rangle)$$

(where $\mathcal{H}$ denotes the homotopy category), and finally that

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{qc,fg}}(Y,\mathcal{Y})}(\mathrm{For}\,\mathcal{F},\mathrm{For}\,\mathcal{G}) \;\cong\; \mathrm{Hom}_{\mathcal{H}^{\mathrm{qc,fg}}(Y,\mathcal{Y})}(\mathrm{For}\,\mathcal{F},\mathrm{For}\,\mathcal{G}).$$

The result follows, since it is clear that

$$\mathrm{Hom}_{\mathcal{H}^{\mathrm{qc,fg}}(Y,\mathcal{Y})}(\mathrm{For}\,\mathcal{F},\mathrm{For}\,\mathcal{G}) \;\cong\; \bigoplus_{m}\mathrm{Hom}_{\mathcal{H}^{\mathrm{fg}}\left(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(Y,\mathcal{Y})\right)}(\mathcal{F},\mathcal{G}\langle m\rangle).$$

Now, let us deduce (ii) from (i). First, by Lemma 6.6.2(i) we can assume $\mathcal{F}$ and $\mathcal{G}$ are locally finitely generated. Let us prove that for any $m\in\mathbb{Z}$ the morphism

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{fg}}\left(\mathcal{C}^{\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{Z}(Y,\widehat{\mathcal{Y}})\right)}(\mathcal{F},\mathcal{G}\langle m\rangle) \;\to\; \mathrm{Hom}_{\mathcal{D}^{\mathrm{fg}}\left(\mathcal{C}^{\mathrm{qc}}_{Z}(Y,\widehat{\mathcal{Y}})\right)}(\mathrm{For}\,\mathcal{F},\mathrm{For}\,\mathcal{G}) \tag{6.6.4}$$

is injective. It is sufficient to prove that if $f:\mathcal{F}\to\mathcal{G}$ is a morphism of $\mathbb{G}_{\mathbf{m}}$-equivariant dg-modules such that $\mathrm{For}(f)=0$ in $\mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc}}_{Z}(Y,\widehat{\mathcal{Y}})\big)$, then $f=0$ in $\mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{Z}(Y,\widehat{\mathcal{Y}})\big)$. By standard properties of localizations of triangulated categories, and using a non-equivariant analogue of Lemma 6.6.2(i), there exists $\mathcal{P}$ in $\mathcal{C}^{\mathrm{qc,fg}}_{Z}(Y,\widehat{\mathcal{Y}})$ and a quasi-isomorphism $\mathcal{G}\xrightarrow{\mathrm{qis}}\mathcal{P}$

whose composition with f is homotopic to 0. The dg-modules $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{P}$ live on a certain infinitesimal neighborhood $Z^{[i]}$ of Z in Y. Applying the injectivity statement in (i) to the scheme $Z^{[i]}$, endowed with the dg-algebra $\widehat{\mathcal{Y}}|_{Z^{[i]}}$, and to the morphism induced by f, we obtain that we can choose $\mathcal{P}$ and the quasi-isomorphism $\mathcal{G} \to \mathcal{P}$ to be $\mathbb{G}_{\mathbf{m}}$-equivariant. This proves the injectivity of (6.6.4).

The injectivity of the morphism in the statement of the lemma follows from the injectivity of (6.6.4), using the fact that the mutliplicative group $\mathbb{G}_{\mathbf{m}}$ acts naturally on the vector space $\mathrm{Hom}_{\mathcal{D}^{\mathrm{fg}}(\mathcal{C}^{\mathrm{qc}}_Z(Y,\widehat{\mathcal{Y}}))}(\mathrm{For}\ \mathcal{F}, \mathrm{For}\ \mathcal{G})$, and that for this action the image of $\mathrm{Hom}_{\mathcal{D}^{\mathrm{fg}}(\mathcal{C}^{\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_Z(Y,\widehat{\mathcal{Y}}))}(\mathcal{F}, \mathcal{G}\langle m\rangle)$ has weight m.

The surjectivity can be proved by similar methods. □

6.7 Proof of Theorem 6.3.4

We have seen in Lemma 5.2.3 that there exists an equivalence of categories

$$\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \cong \mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_{\mathbf{m}}}(\mathcal{B}^{(1)}, \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \tag{6.7.1}$$

where the internal grading on $\pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}$ is induced by the action of $\mathbb{G}_{\mathbf{m}}$ defined in (5.2.1), and $\mathfrak{g}^{(1)}$ is in bidegree $(-1, 2)$.

In this section we consider $\widehat{\mathcal{B}^{(1)}}$ as the formal neighborhood of the zero section in $\widetilde{\mathfrak{g}}^{(1)}$. We have seen in 6.3 that the completion $\widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}}$, considered as a coherent sheaf of rings on $\widehat{\mathcal{B}^{(1)}} \subset \widetilde{\mathfrak{g}}^{(1)}$, is endowed with a $\mathbb{G}_{\mathbf{m}}$-equivariant structure, compatible with that of $\widetilde{\mathfrak{g}}^{(1)}$. Hence we can consider the category

$$\mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big)$$

as in 6.6 (situation (b)). Now we have:

Lemma 6.7.2. *There exists an equivalence of categories*

$$\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_{\mathbf{m}}}(\mathcal{B}^{(1)}, \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \cong \mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big).$$

Proof. By Lemma 6.6.1, there exists an equivalence of categories

$$\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_{\mathbf{m}}}(\mathcal{B}^{(1)}, \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \cong \mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(\mathcal{B}^{(1)}, \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big).$$

As π is affine, the functor π_* induces an equivalence of categories

$$\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathfrak{g}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \to \mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(\mathcal{B}^{(1)}, \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})).$$

Thus, composing the inverse of this equivalence with the previous one, we obtain an equivalence

$$\mathcal{D}^{\mathrm{qc,fg}}_{\mathbb{G}_{\mathbf{m}}}(\mathcal{B}^{(1)}, \pi_* \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \cong \mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathfrak{g}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big)$$

Now, using the fact that any object of $\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathfrak{g}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$ has its cohomology supported on $\mathcal{B}^{(1)}$, we obtain by Lemma 6.6.2(ii) an equivalence

$$\mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc}}_{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathfrak{g}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big) \;\cong\; \mathcal{D}\big(\mathcal{C}^{\mathrm{qc,fg},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big).$$

Then, using $\mathbb{G}_{\mathbf{m}}$-equivariant analogues of the functors F and G of the proof of Theorem 3.3.3, we obtain an equivalence

$$\mathcal{C}^{\mathrm{qc,fg},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}, \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \;\cong\; \mathcal{C}^{\mathrm{qc,fg},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})).$$

The equivalence of the lemma follows from all these equivalences, and again Lemma 6.6.2(i). □

We have seen in 6.3 that the completion $(\mathcal{U}\mathfrak{g})^{\hat{0}}_{\hat{0}}$, *i.e.* the restriction of the sheaf of algebras $\mathcal{U}\mathfrak{g}$ on $\mathrm{Spec}(\mathfrak{Z})$ to the formal neighborhood of $(0,0)$, considered as a sheaf of algebras on the formal neighborhood[9] $\widehat{\{0\}}$ of $\{0\}$ in $\mathfrak{g}^{*(1)}$, is endowed with a $\mathbb{G}_{\mathbf{m}}$-equivariant structure, compatible with that of $\mathfrak{g}^{*(1)}$. Hence we are again in situation (b) of 6.6. We simplify the notation for the categories of sheaves of $\mathcal{U}\mathfrak{g}$-modules, and denote *e.g.* by $\mathcal{C}^{\mathrm{fg},\mathbb{G}_{\mathbf{m}}}_{(0,0)}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$ the category $\mathcal{C}^{\mathrm{qc,fg},\mathbb{G}_{\mathbf{m}}}_{\{0\}}(\mathfrak{g}^{*(1)}, \mathcal{U}\mathfrak{g}_{|\widehat{\{0\}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$. By Lemma 6.6.2(i) we have an equivalence of categories

$$\mathcal{D}\big(\mathcal{C}^{\mathrm{fg},\mathbb{G}_{\mathbf{m}}}_{(0,0)}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big) \;\cong\; \mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathbb{G}_{\mathbf{m}}}_{(0,0)}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big). \tag{6.7.3}$$

Recall the remarks before Lemma 3.3.5. Let us consider the following forgetful functors (of the internal grading):

$$\mathrm{For}: \mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathbb{G}_{\mathbf{m}}}_{(0,0)}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big) \;\to\; \mathcal{D}^{\mathrm{fg}}_0(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})),$$
$$\mathrm{For}: \mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big) \;\to\; \mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big).$$

Clearly, the category $\mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big)$ is equivalent to the triangulated category $\mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}\times\{0\}}(X, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$ (see *e.g.* Proposition 3.2.4). Here $X = \widetilde{\mathfrak{g}}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$, as in subsection 3.3.

By Lemma 6.6.3(ii) we have:

Lemma 6.7.4. (i) *For M, N in $\mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathbb{G}_{\mathbf{m}}}_{(0,0)}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big)$ there is an isomorphism*

$$\bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^{\mathrm{fg}}\left(\mathcal{C}^{\mathbb{G}_{\mathbf{m}}}_{(0,0)}(\mathcal{U}\mathfrak{g}\otimes_{\Bbbk}\Lambda(\mathfrak{g}^{(1)}))\right)}(M, N\langle m\rangle) \;\cong\; \mathrm{Hom}_{\mathcal{D}^{\mathrm{fg}}_0(\mathcal{U}\mathfrak{g}\otimes_{\Bbbk}\Lambda(\mathfrak{g}^{(1)}))}(\mathrm{For}\, M, \mathrm{For}\, N).$$

(ii) *For $\mathcal{F}, \mathcal{G}$ in $\mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big)$ there is an isomorphism*

$$\bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^{\mathrm{fg}}\left(\mathcal{C}^{\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}}\otimes_{\Bbbk}\Lambda(\mathfrak{g}^{(1)}))\right)}(\mathcal{F}, \mathcal{G}\langle m\rangle)$$
$$\cong \quad \mathrm{Hom}_{\mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}\times\{0\}}(X, \widetilde{\mathcal{D}}\otimes_{\Bbbk}\Lambda(\mathfrak{g}^{(1)}))}(\mathrm{For}\,\mathcal{F}, \mathrm{For}\,\mathcal{G}).$$

[9] This formal neighborhood is also isomorphic to the formal neiborhood of $\{(0,0)\}$ in $\mathrm{Spec}(\mathfrak{Z}) \cong \mathfrak{g}^{*(1)} \times_{\mathfrak{h}^{*(1)}/W} \mathfrak{h}^*/(W, \bullet)$. We will not distinguish these two formal neighborhoods.

Corollary 6.7.5. *There exists a fully faithful functor*

$$R\Gamma_{\mathbb{G}_{\mathbf{m}}} : \mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)},\, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big) \;\to\; \mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathbb{G}_{\mathbf{m}}}_{(0,0)}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big).$$

Proof. Let us denote by $\mathcal{C}^{+,\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)},\, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$ the full subcategory of the category $\mathcal{C}^{\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$ consisting of bounded below objects. It is clear from the definition (using a truncation functor) that, with obvious notation, we have an equivalence of categories

$$\mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{+,\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)},\, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big) \;\cong\; \mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)},\, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big).$$

We denote by Γ^+ the functor

$$\mathcal{C}^{+,\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)},\, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \;\to\; \mathcal{C}^{\mathbb{G}_{\mathbf{m}}}_{(0,0)}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$$

induced by the global sections functor $\Gamma(\widetilde{\mathfrak{g}}^{(1)}, -)$. Let us first show that the derived functor $R\Gamma^+$ (in the sense of Deligne) is defined everywhere, *i.e.* that every object of $\mathcal{C}^{+,\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)},\, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$ has a right resolution which is split on the right (see 1.4).

Every object $\mathcal{F}$ of $\mathcal{C}^{+,\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)},\, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$ has a resolution $\mathcal{F} \xrightarrow{\mathrm{qis}} \mathcal{I}$ where $\mathcal{I}$ is in $\mathcal{C}^{+,\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)},\, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$, and each $\mathcal{I}^p$ ($p \in \mathbb{Z}$) is acyclic for the functor $\Gamma(\widetilde{\mathfrak{g}}^{(1)}, -) : \mathrm{QCoh}(\widetilde{\mathfrak{g}}^{(1)}) \to \mathrm{Vect}(\Bbbk)$. Indeed, let $\widetilde{\mathfrak{g}}^{(1)} = \bigcup_{\alpha=1}^{n} X_\alpha$ be an affine open covering such that each X_α is $\mathbb{G}_{\mathbf{m}}$-stable (*e.g.* the inverse image of an affine open covering of $\mathcal{B}^{(1)}$). For each α, let $j_\alpha : X_\alpha \hookrightarrow X$ be the inclusion. Then there is an inclusion

$$\mathcal{F} \;\hookrightarrow\; \bigoplus_{\alpha=1}^{n} (j_\alpha)_*(j_\alpha)^*\mathcal{F}.$$

Doing the same construction for the cokernel of this inclusion, repeating, and finally taking a total complex, as *e.g.* in the proof of Lemma 1.3.7, one obtains the resolution $\mathcal{I}$. Such a resolution is clearly split on the right for the functor Γ^+.

By this construction, it is clear that the following diagram is commutative, where the vertical arrows are the natural forgetful functors, and the bottom horizontal arrow is the functor considered in (3.3.10):

$$\begin{tikzcd}
\mathcal{D}\big(\mathcal{C}^{+,\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)},\, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big) \arrow[r, "R\Gamma^+"] \arrow[d, "\mathrm{For}"'] & \mathcal{D}\big(\mathcal{C}^{\mathbb{G}_{\mathbf{m}}}_{(0,0)}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}(X,\, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \arrow[r, "R\Gamma"] & \mathcal{D}(\mathrm{Spec}(\Bbbk),\, \mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})).
\end{tikzcd}$$

It follows from this diagram and the results just below (3.3.10) that the functor $R\Gamma^+$ restricts to a functor

$$R\Gamma_{\mathbb{G}_{\mathbf{m}}} : \mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{+,\mathrm{qc},\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)},\, \widetilde{\mathcal{D}}_{|\widehat{\mathcal{B}^{(1)}}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big) \;\to\; \mathcal{D}^{\mathrm{fg}}\big(\mathcal{C}^{\mathbb{G}_{\mathbf{m}}}_{(0,0)}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big),$$

which corresponds to the functor

$$R\Gamma : \mathcal{D}^{\mathrm{qc,fg}}_{\mathcal{B}^{(1)}\times\{0\}}(X, \widetilde{\mathcal{D}} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})) \ \to\ \mathcal{D}^{\mathrm{fg}}_0(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))$$

of (3.3.11) under the natural forgetful functors. We have proved in the course of the proof of Theorem 3.3.3 that the latter functor is fully faithful (and even an equivalence of categories, but this is not needed here). It follows easily, using Lemma 6.7.4, that the functor $R\Gamma_{\mathbb{G}_{\mathbf{m}}}$ is also fully faithful. This concludes the proof of Corollary 6.7.5. □

Thus, using equivalence (6.7.1), Lemma 6.7.2, Corollary 6.7.5 and equivalence (6.7.3), we obtain a fully faithful functor

$$\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^*\times\mathcal{B}} \mathcal{B})^{(1)}) \ \to\ \mathcal{D}\big(\mathcal{C}^{\mathrm{fg},\mathbb{G}_{\mathbf{m}}}_{(0,0)}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big).$$

Hence to finish the proof of Theorem 6.3.4 we finally only have to prove the following lemma.

Lemma 6.7.6. *There exists an equivalence of categories*

$$\mathcal{D}\big(\mathcal{C}^{\mathrm{fg},\mathbb{G}_{\mathbf{m}}}_{(0,0)}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big) \ \cong\ \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg,gr}}_0((\mathcal{U}\mathfrak{g})_0).$$

Proof. The natural morphism

$$\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}) \twoheadrightarrow (\mathcal{U}\mathfrak{g})_0 \twoheadrightarrow (\mathcal{U}\mathfrak{g})^{\hat{0}}_0$$

induces a functor (restriction of scalars):

$$\Psi : \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg,gr}}_0((\mathcal{U}\mathfrak{g})_0) \ \to\ \mathcal{D}\big(\mathcal{C}^{\mathrm{fg},\mathbb{G}_{\mathbf{m}}}_{(0,0)}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big).$$

This functor corresponds to the functor considered in (3.3.2) under the natural forgetful functors. We deduce, as in the proof of Corollary 6.7.5, that Ψ is fully faithful.

Now we prove that it is essentially surjective. More precisely, we prove that every object M of $\mathcal{D}\big(\mathcal{C}^{\mathrm{fg},\mathbb{G}_{\mathbf{m}}}_{(0,0)}(\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)}))\big)$ is in the essential image of Ψ by induction on $l(M)$ where, as in the proof of Lemma 3.2.2, $l(M) = -1$ if $M \cong 0$ and, for $M \neq 0$,

$$l(M) := \max\{i \in \mathbb{Z} \mid H^i(M) \neq 0\} - \min\{i \in \mathbb{Z} \mid H^i(M) \neq 0\}.$$

The result is clear if $l(M) = -1$. If $l(M) = 0$, then M is quasi-isomorphic to a $\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \Lambda(\mathfrak{g}^{(1)})$-dg-module N concentrated in one cohomological degree. It follows easily from the definitions that N is a restricted $\mathcal{U}\mathfrak{g}$-module. Hence it is in the image of Ψ.

Now let $n > 0$, and assume that any N with $l(N) < n$ is in the image of Ψ. Let M such that $l(M) = n$, and let j be the lowest integer such that $H^j(M) \neq 0$. We can assume that $M^k = 0$ for $k < j$. Let $M' := \ker(d^j_M)$, considered as a complex concentrated in degree j, and $M'' := \mathrm{Coker}(M' \to M)$. We have $l(M') = 0$, hence M' is in the image of Ψ. Let P' be such that $M' = \Psi(P')$. By induction, there exists P'' in $\mathcal{D}^b\mathrm{Mod}^{\mathrm{fg,gr}}_0((\mathcal{U}\mathfrak{g})_0)$ such that $\Psi(P'') \cong M''$. As Ψ is fully faithful, the natural morphism $M'' \to M'[1]$ comes from a morphism $P'' \to P'[1]$ in $\mathcal{D}^b\mathrm{Mod}^{\mathrm{fg,gr}}_0((\mathcal{U}\mathfrak{g})_0)$. Then if P is the cone of this morphism, standard properties of triangulated categories ensure that $M \cong \Psi(P[-1])$. This finishes the proof of Lemma 6.7.6. □

The proof of Theorem 6.3.4 is now complete.

7 Simple $(\mathcal{U}\mathfrak{g})^0$-modules

In this section we study in more details the left hand side of diagram ($*$) after Proposition 3.3.14.

7.1 The "semi-simple" functors $\mathfrak{S}_\delta$

Let $\alpha \in \Phi$ be a finite simple root. Recall the subvariety $Y_\alpha \subset \widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}$ defined in subsection 5.3 (see also subsection II.7.1). We denote by $\mathfrak{S}_\alpha$ the convolution functor

$$F^{\mathcal{O}_{Y_\alpha^{(1)}}(-\rho,\rho-\alpha)}_{\widetilde{\mathcal{N}}^{(1)} \to \widetilde{\mathcal{N}}^{(1)}} : \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)}) \ \to \ \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)}).$$

Now let $\alpha_0 \in \Phi_{\mathrm{aff}} - \Phi$. Recall the notation β, b_0 of Lemma 6.1.2, and the notation for the B'_{aff}-actions in subsection 5.1. We define

$$\mathfrak{S}_{\alpha_0} := \mathbf{K}_{b_0} \circ \mathfrak{S}_\beta \circ \mathbf{K}_{(b_0)^{-1}}.$$

These functors stabilize the subcategory $\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$. They will be related in 8.2 to the reflection functors of 6.1.

For all $\delta \in \Phi_{\mathrm{aff}}$ we have an exact triangle of endofunctors of $\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)})$:

$$\mathfrak{S}_\delta \to \mathbf{K}_{C(s_\delta)} \to \mathrm{Id}. \tag{7.1.1}$$

For $\delta \in \Phi$, this follows from the exact sequence (5.3.4), using the fact that $C(s_\delta) = T_\delta$. For $\delta = \alpha_0$, this is the conjugate of the corresponding triangle for β, using the relation $C(s_0) = b_0 C(s_\beta)(b_0)^{-1}$.

We give a representation-theoretic interpretation of these functors in Proposition 7.1.2. Recall the equivalence

$$\epsilon_0^{\mathcal{B}} : \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \ \xrightarrow{\sim} \ \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0)$$

of equation (1.2.3) in chapter I. We have defined the objects $\mathcal{L}_w$ in subsection 4.4.

Proposition 7.1.2. *Let $w \in W^0$, and $\delta \in \Phi_{\mathrm{aff}}$ be such that $ws_\delta \bullet 0 > w \bullet 0$. Recall the $(\mathcal{U}\mathfrak{g})^0_0$-module $Q_\delta(w)$ defined in* (5.5.2). *We have*

$$\mathfrak{S}_\delta \mathcal{L}_w \ \cong \ (\epsilon_0^{\mathcal{B}})^{-1}(Q_\delta(w)).$$

Proof. The exact triangle of functors (7.1.1) induces an exact triangle in $\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$:

$$\mathfrak{S}_\delta(\mathcal{L}_w) \to \mathbf{K}_{C(s_\delta)}(\mathcal{L}_w) \to \mathcal{L}_w. \tag{7.1.3}$$

Let $i : \widetilde{\mathcal{N}} \hookrightarrow \widetilde{\mathfrak{g}}$ be the inclusion. Then we have $i_* \circ \mathbf{K}_{C(s_\delta)} \cong \mathbf{J}_{C(s_\delta)} \circ i_*$ (see subsection 5.1). Hence triangle (7.1.3) induces an exact triangle

$$\gamma_0^{\mathcal{B}} \circ i_* \circ \mathfrak{S}_\delta(\mathcal{L}_w) \to \gamma_0^{\mathcal{B}} \circ \mathbf{J}_{C(s_\delta)} \circ i_*(\mathcal{L}_w) \to \gamma_0^{\mathcal{B}} \circ i_*(\mathcal{L}_w). \tag{7.1.4}$$

By construction we have an isomorphism of functors $\gamma_0^{\mathcal{B}} \circ i_* \cong \mathrm{Incl} \circ \epsilon_0^{\mathcal{B}}$, where Incl is induced by the inclusion $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0) \hookrightarrow \mathrm{Mod}_{(0,0)}^{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$. In particular, $L(w \bullet 0) \cong \gamma_0^{\mathcal{B}} \circ i_*(\mathcal{L}_w)$. Also, using diagram (5.1.1), we have

$$\gamma_0^{\mathcal{B}} \circ \mathbf{J}_{C(s_\delta)} \circ i_*(\mathcal{L}_w) \;\cong\; \mathbf{I}_{C(s_\delta)} \circ \gamma_0^{\mathcal{B}} \circ i_*(\mathcal{L}_w) \;\cong\; \mathbf{I}_{C(s_\delta)}(L(w \bullet 0)).$$

Hence triangle (7.1.4) induces an exact triangle

$$\mathrm{Incl} \circ \epsilon_0^{\mathcal{B}} \circ \mathfrak{S}_\delta(\mathcal{L}_w) \to \mathbf{I}_{C(s_\delta)}(L(w \bullet 0)) \to L(w \bullet 0). \tag{7.1.5}$$

Now by definition (see [BMR06, 2.3]), $\mathbf{I}_{C(s_\delta)}(L(w \bullet 0))$ is the cone of the natural morphism $L(w \bullet 0) \to R_\delta L(w \bullet 0)$. This morphism is the morphism ϕ_δ^w of subsection 5.5, hence $\mathbf{I}_{C(s_\delta)}(L(w \bullet 0)) \cong \mathrm{Coker}(\phi_\delta^w)$. Moreover, under this identification, the second morphism in (7.1.5) is induced by ψ_δ^w (again with the notation of 5.5). Hence triangle (7.1.5) induces an isomorphism $\mathrm{Incl} \circ \epsilon_0^{\mathcal{B}} \circ \mathfrak{S}_\delta(\mathcal{L}_w) \cong Q_\delta(w)$. It follows that $\epsilon_0^{\mathcal{B}} \circ \mathfrak{S}_\delta(\mathcal{L}_w)$ has cohomology only in degree 0. As the restriction of Incl to objects having cohomology only in degree 0 is fully faithful, the result follows. □

To finish this subsection, let us remark that for all $\delta \in \Phi_{\mathrm{aff}}$ there is a natural functor

$$\mathfrak{S}_\delta^{\mathbb{G}_\mathbf{m}} : \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(\widetilde{\mathcal{N}}^{(1)}) \;\to\; \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(\widetilde{\mathcal{N}}^{(1)})$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, "\mathfrak{S}_\delta^{\mathbb{G}_\mathbf{m}}"] \arrow[d, "\mathrm{For}"] & \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, "\mathfrak{S}_\delta"] & \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)}),
\end{tikzcd} \tag{7.1.6}$$

namely the graded convolution with kernel $\mathcal{O}_{Y_\delta^{(1)}}(-\rho, \rho-\delta)$ (with its natural $\mathbb{G}_\mathbf{m}$-structure) if $\delta \in \Phi$, or the conjugate of the convolution with kernel $\mathcal{O}_{Y_\beta^{(1)}}(-\rho, \rho-\beta)$ by $\mathbf{K}_{b_0}^{\mathrm{gr}}$ if $\delta = \alpha_0$.

7.2 Graded $(\mathcal{U}\mathfrak{g})^0$-modules

As in subsection 6.3, we have (see [BMR06, 3.4.1]):

$$(\mathcal{U}\mathfrak{g})^0 \;\cong\; R\Gamma(\widetilde{\mathcal{N}}^{(1)}, \widetilde{\mathcal{D}}_{|\widetilde{\mathcal{N}}^{(1)} \times \{0\}}).$$

We have defined an action of $\mathbb{G}_\mathbf{m}$ on $\widetilde{\mathcal{N}}^{(1)}$ in 6.3 (note that it is not the restriction of the action on $\widetilde{\mathfrak{g}}^{(1)}$, but its composition with $t \mapsto t^{-1}$). The same arguments as in 6.3 show that there exists a $\mathbb{G}_\mathbf{m}$-equivariant structure on the algebra $(\mathcal{U}\mathfrak{g})_{\hat{0}}^0$ (the completion of $(\mathcal{U}\mathfrak{g})^0$ with respect to the image of the augmentation ideal of $\mathfrak{Z}_{\mathrm{Fr}}$ corresponding to the character $0 \in \mathfrak{g}^{*(1)}$), compatible with the $\mathbb{G}_\mathbf{m}$-structure on $\mathfrak{g}^{*(1)}$ induced by the action on $\widetilde{\mathcal{N}}^{(1)}$. We denote by $\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})^0)$ the category of graded $(\mathcal{U}\mathfrak{g})^0$-modules with trivial generalized

Frobenius central character (these modules are modules over the quotient of $(\mathcal{U}\mathfrak{g})^0$ by a power of the ideal generated by $\mathfrak{g}^{(1)}$; this quotient is a graded algebra, hence we can speak of *graded* modules). Arguments similar to (and easier than) the ones of subsection 6.7 prove the following theorem, which is a "graded version" of equivalence (1.2.3) in Theorem I.1.2.1(i):

Theorem 7.2.1. *There exists a fully faithful functor*

$$\widetilde{\epsilon}_0^{\mathcal{B}} : \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}^{(1)}) \ \to \ \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})^0),$$

commuting with the internal shifts $\langle 1 \rangle$, and such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, "\widetilde{\epsilon}_0^{\mathcal{B}}"] \arrow[d, "\mathrm{For}"] & \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})^0) \arrow[d, "\mathrm{For}"] \\ \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, "\epsilon_0^{\mathcal{B}}"] & \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0). \end{tikzcd}$$

Now, consider the category $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0)$. Using again Theorem 5.6.1, each simple module $L(w\bullet 0)$ (for $w \in W^0$) can be lifted to a graded module $L^{\mathrm{gr}}(w\bullet 0)$ in $\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})^0)$ (uniquely, up to isomorphism and to internal shift). Here the algebra $(\mathcal{U}\mathfrak{g})_{\hat{0}}^0$ is not finite dimensional, but it acts on simple modules through $(\mathcal{U}\mathfrak{g})_0^0$, which is finite dimensional, hence we can still apply Theorem 5.6.1. In this subsection we fix an arbitrary choice for these lifts.

Recall that we denote by $i : \widetilde{\mathcal{N}}^{(1)} \hookrightarrow \widetilde{\mathfrak{g}}^{(1)}$ the natural inclusion. Let also $j : \mathcal{B}^{(1)} \hookrightarrow \widetilde{\mathcal{N}}^{(1)}$, $k : \mathcal{B}^{(1)} \hookrightarrow \widetilde{\mathfrak{g}}^{(1)}$ be the inclusions of the zero-sections. Recall Lemma I.1.4.1. We deduce the following corollary, which generalizes some of the computations of sections I.2 and I.3.

Corollary 7.2.2. *Let $\omega \in W'_{\mathrm{aff}}$ such that $\ell(\omega) = 0$. Write $\omega = w \cdot t_\mu$ ($\mu \in \mathbb{X}$, $w \in W$). Then we have*

$$j_*\mathcal{O}_{\mathcal{B}^{(1)}}(-\rho + \mu)[\ell(w)] \ \cong \ \mathcal{L}_\omega.$$

Proof. By Lemma I.1.4.1,

$$\epsilon_0^{\mathcal{B}}(j_*\mathcal{O}_{\mathcal{B}^{(1)}}(-\rho + \mu)) \ \cong \ R\Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(p\mu)).$$

By hypothesis, $\omega \bullet 0 = w \bullet (p\mu)$. Hence $w^{-1} \bullet (\omega \bullet 0) = p\mu$. Using Borel-Weil-Bott theorem ([Jan03, II.5.5-6]), we deduce

$$\epsilon_0^{\mathcal{B}}(j_*\mathcal{O}_{\mathcal{B}^{(1)}}(-\rho + \mu)[\ell(w)]) \ \cong \ \mathrm{Ind}_B^G(\omega \bullet 0) \ \cong \ L(\omega \bullet 0).$$

This concludes the proof. □

Proposition 7.2.3. *For all $w \in W^0$, $L^{\mathrm{gr}}(w \bullet 0)$ is in the essential image of the functor $\widetilde{\epsilon}_0^{\mathcal{B}}$.*

Proof. This proof is similar to that of Proposition 6.5.1. We use an ascending induction on $\ell(w)$. For $\ell(w)=0$, by Corollary 7.2.2 we have $\mathcal{L}_w \cong j_* \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho+\mu)[\ell(v)]$ where $w = v \cdot t_\mu$ ($v \in W$, $\mu \in \mathbb{X}$). Clearly, $j_* \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho+\mu)$ has a structure of a $\mathbb{G}_\mathbf{m}$-equivariant coherent sheaf, hence can be considered as an object of $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$. By Theorem 5.6.1, the image of this object under $\widetilde{\epsilon}_0^{\mathcal{B}}$ is isomorphic to $L^{\mathrm{gr}}(w \bullet 0)$, up to a shift. As the functor $\widetilde{\epsilon}_0^{\mathcal{B}}$ commutes with the internal shifts, the result follows when $\ell(w)=0$.

Now assume the result is true when $\ell(w) < n$, and let $w \in W^0$ such that $\ell(w)=n$. Let $\delta \in \Phi_{\mathrm{aff}}$ be such that $ws_\delta \in W^0$ and $\ell(ws_\delta) < \ell(w)$. By induction there exists $\mathcal{L}^{\mathrm{gr}}$ in $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$ such that $\widetilde{\epsilon}_0^{\mathcal{B}}(\mathcal{L}^{\mathrm{gr}}) \cong L^{\mathrm{gr}}(ws_\delta \bullet 0)$. Then, by diagram (7.1.6) and Proposition 7.1.2, the image under the forgetful functor

$$\mathrm{For} : \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})^0) \ \to \ \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0)$$

of the object $\widetilde{\epsilon}_0^{\mathcal{B}}(\mathfrak{S}_\delta^{\mathbb{G}_\mathbf{m}}\mathcal{L}^{\mathrm{gr}})$ is $Q_\delta(ws_\delta)$. By Theorem 5.5.3, we have $Q_\delta(ws_\delta) \cong L(w \bullet 0) \oplus N$ where N is a direct sum of modules of the form $L^{\mathrm{gr}}(v \bullet 0)$ with $\ell(v) < \ell(w)$. Hence, by Corollary 5.6.4(ii) and its proof, we have $\widetilde{\epsilon}_0^{\mathcal{B}}(\mathfrak{S}_\delta^{\mathbb{G}_\mathbf{m}}\mathcal{L}^{\mathrm{gr}}) \cong L^{\mathrm{gr}}(w \bullet 0)\langle i \rangle \oplus N^{\mathrm{gr}}$ for some $i \in \mathbb{Z}$, where N^{gr} is a direct sum of modules of the form $L(v \bullet 0)\langle j \rangle$ with $\ell(v) < \ell(w)$, $j \in \mathbb{Z}$. By induction hypothesis, N^{gr} is in the essential image of $\widetilde{\epsilon}_0^{\mathcal{B}}$. We conclude as in the proof of Proposition 6.5.1 that $L^{\mathrm{gr}}(w \bullet 0)$ is also in this image. □

Remark 7.2.4. It follows easily from Proposition 7.2.3 that the functor $\widetilde{\epsilon}_0^{\mathcal{B}}$ is essentially surjective. Hence it is an equivalence of categories.

7.3 Dg versions of the functors $\mathfrak{S}_\delta$

Let $\alpha \in \Phi$ be a finite simple root. Let P_α be the parabolic subgroup of G containing B associated to $\{\alpha\}$, let $\mathfrak{p}_\alpha$ be its Lie algebra, and let $\mathcal{P}_\alpha = G/P_\alpha$ be the associated partial flag variety. We define the variety

$$\widetilde{\mathcal{N}}_\alpha := T^*\mathcal{P}_\alpha = \{(X, gP_\alpha) \in \mathfrak{g}^* \times \mathcal{P}_\alpha \mid X_{|g \cdot \mathfrak{p}_\alpha} = 0\}. \tag{7.3.1}$$

There exists a natural injection

$$j_\alpha : (\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)} \hookrightarrow \widetilde{\mathcal{N}}^{(1)}.$$

We also denote by

$$\rho_\alpha : (\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)} \to \widetilde{\mathcal{N}}_\alpha^{(1)}$$

the morphism defined by base change.

Consider the following diagram:

$$\begin{tikzcd}
 & & (\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B}) \times_{\widetilde{\mathcal{N}}_\alpha} (\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B}) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
 & \widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B} \arrow[dl, hook', "j_\alpha"'] \arrow[dr, "\rho_\alpha"] & & \widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B} \arrow[dl, "\rho_\alpha"'] \arrow[dr, hook, "j_\alpha"] & \\
\widetilde{\mathcal{N}} & & \widetilde{\mathcal{N}}_\alpha & & \widetilde{\mathcal{N}}.
\end{tikzcd}$$

Here to save space we have omitted the Frobenius twists. The flat base change theorem (see [Har77, II.5.12]) implies that we have an isomorphism of functors from $\mathcal{D}^b\mathrm{Coh}((\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)})$ to itself:

$$L(\rho_\alpha)^* \circ R(\rho_\alpha)_* \;\cong\; R(p_2)_* \circ L(p_1)^*. \tag{7.3.2}$$

Moreover, the variety $(\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B}) \times_{\widetilde{\mathcal{N}}_\alpha} (\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})$ is isomorphic to the subvariety Y_α of $\widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}$. For $\lambda \in \mathbb{X}$, we denote by Shift_λ the tensor product with $\mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}(\lambda)$. Then we have

$$\begin{aligned}
\mathrm{Shift}_{-\rho} \circ \mathfrak{S}_\alpha \circ \mathrm{Shift}_\rho \;&\cong\; \mathrm{Shift}_{-\rho} \circ F^{\mathcal{O}_{Y_\alpha^{(1)}}(-\rho,\rho-\alpha)}_{\widetilde{\mathcal{N}}^{(1)} \to \widetilde{\mathcal{N}}^{(1)}} \circ \mathrm{Shift}_\rho \\
&\cong\; \mathrm{Shift}_{-\alpha} \circ F^{\mathcal{O}_{Y_\alpha^{(1)}}}_{\widetilde{\mathcal{N}}^{(1)} \to \widetilde{\mathcal{N}}^{(1)}} \\
&\cong\; \mathrm{Shift}_{-\alpha} \circ (R(j_\alpha)_* \circ R(p_2)_* \circ L(p_1)^* \circ L(j_\alpha)^*) \\
&\cong\; \mathrm{Shift}_{-\alpha} \circ (R(j_\alpha)_* \circ L(\rho_\alpha)^* \circ R(\rho_\alpha)_* \circ L(j_\alpha)^*).
\end{aligned}$$

Here the last isomorphism is given by (7.3.2). Now recall the constructions of section 2. In Corollary 2.5.3 we have constructed functors associated to j_α:

$$\begin{aligned}
R(\widetilde{j}_{\alpha\mathbb{G}_\mathbf{m}})_* : \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)}) \;&\to\; \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}), \\
L(\widetilde{j}_{\alpha\mathbb{G}_\mathbf{m}})^* : \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) \;&\to\; \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)}).
\end{aligned}$$

Similarly, in Corollary 2.4.5 we have constructed functors associated to ρ_α:

$$\begin{aligned}
R(\widetilde{\rho_{\alpha\mathbb{G}_\mathbf{m}}})_* : \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)}) \;&\to\; \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}_\alpha^{(1)}), \\
L(\widetilde{\rho_{\alpha\mathbb{G}_\mathbf{m}}})^* : \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}_\alpha^{(1)}) \;&\to\; \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)}).
\end{aligned}$$

We define the functor

$$\mathfrak{S}_\alpha^{\mathrm{gr}} : \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) \;\to\; \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}),$$

which sends the object $\mathcal{M}$ to

$$\mathcal{O}_{\mathcal{B}^{(1)}}(\rho-\alpha) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \big(R(\widetilde{j}_{\alpha\mathbb{G}_\mathbf{m}})_* \circ L(\widetilde{\rho_{\alpha\mathbb{G}_\mathbf{m}}})^* \\ \circ R(\widetilde{\rho_{\alpha\mathbb{G}_\mathbf{m}}})_* \circ L(\widetilde{j}_{\alpha\mathbb{G}_\mathbf{m}})^*(\mathcal{M} \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho))\big).$$

Using Corollaries 2.4.5 and 2.5.3, and the isomorphism above, the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, "\mathfrak{S}_\alpha^{\mathrm{gr}}"] \arrow[d, "\mathrm{For}"'] & \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, "\mathfrak{S}_\alpha"] & \mathcal{D}^b\mathrm{Coh}(\widetilde{\mathcal{N}}^{(1)}).
\end{tikzcd}$$

The following diagrams also commute, where η and ζ are the functors defined in 4.2:

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, "\mathfrak{S}_\alpha^{\mathrm{gr}}"] \arrow[d, "\eta"'] & \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[d, "\eta"] \\
\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, "\mathfrak{S}_\alpha^{\mathbb{G}_\mathbf{m}}"] & \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}(\widetilde{\mathcal{N}}^{(1)}),
\end{tikzcd} \tag{7.3.3}$$

$$\begin{tikzcd} \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, "\mathfrak{S}_\alpha^{\mathbb{G}_\mathbf{m}}"] \arrow[d, "\zeta"'] & \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[d, "\zeta"'] \\ \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, "\mathfrak{S}_\alpha^{\mathrm{gr}}"] & \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}). \end{tikzcd} \tag{7.3.4}$$

Indeed, the commutation of the first diagram follows from the definitions and Lemmas 2.4.4 and 2.5.2, and the commutation of the second one follows from the commutation of the first one.

Now, let us define an action of B'_{aff} on $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$. Recall the Koszul duality $\kappa_{\mathcal{B}}$ defined in (3.1.1). For $b \in B'_{\mathrm{aff}}$ we define

$$\mathbf{K}_b^{\mathrm{gr}} : \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) \ \to \ \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$$

by the formula

$$\mathbf{K}_b^{\mathrm{gr}} := \mathrm{Shift}_\rho \circ \kappa_{\mathcal{B}}^{-1} \circ \mathbf{J}_b^{\mathrm{gr}} \circ \kappa_{\mathcal{B}} \circ \mathrm{Shift}_{-\rho}.$$

Here, Shift_λ denotes the shift by $\mathcal{O}_{\mathcal{B}^{(1)}}(\lambda)$.

Consider the affine simple root $\alpha_0 \in \Phi_{\mathrm{aff}} - \Phi$. Recall the notation b_0, β from Lemma 6.1.2. Then we define the functor

$$\mathfrak{S}_{\alpha_0}^{\mathrm{gr}} := \mathbf{K}_{b_0}^{\mathrm{gr}} \circ \mathfrak{S}_\beta^{\mathrm{gr}} \circ \mathbf{K}_{(b_0)^{-1}}^{\mathrm{gr}}. \tag{7.3.5}$$

It is not clear from this definition that the diagrams analogous to (7.3.3) and (7.3.4) are commutative. We will consider this issue in 8.3.

8 Proof of Theorem 4.4.3

In this section we prove the key-result of our reasoning, namely Theorem 4.4.3.

8.1 Alternative statement of the theorem

First, let us state a version of Theorem 4.4.3 in representation-theoretic terms.

Recall the Koszul duality equivalence of (3.1.1):

$$\kappa_{\mathcal{B}} : \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) \ \xrightarrow{\sim} \ \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}).$$

Recall that the functor $\widetilde{\gamma}_0^{\mathcal{B}}$ of Theorem 6.3.4 is fully faithful, and that its essential image contains the lifts of the projective modules $P(v \bullet 0)$ for $v \in W^0$ (see Proposition 6.5.1). Hence, for any choice of a lift $P^{\mathrm{gr}}(v \bullet 0)$ of $P(v \bullet 0)$ as a graded $(\mathcal{U}\mathfrak{g})_0^{\hat{0}}$-module (this choice is unique up to isomorphism and internal shift), there exists an object[10] $\mathcal{P}_v^{\mathrm{gr}}$ of $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$, unique up to isomorphism, such that

$$P^{\mathrm{gr}}(v \bullet 0) \cong \widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{P}_v^{\mathrm{gr}}).$$

[10]As observed in subsection 4.4, this object does not depend on the choice $\lambda = 0$. For this reason, 0 does not appear in the notation.

The same applies to the functor $\widetilde{\epsilon}_0^{\mathcal{B}}$ of Theorem 7.2.1, replacing the projective modules by the simple modules $L(v \bullet 0)$ (see Proposition 7.2.3).

Theorem 4.4.3 is clearly equivalent to the following statement, which we will refer to as statement (‡). This is the statement we will prove in 8.4.

Assume $p > h$ is large enough so that Lusztig's conjecture is true.

There is a unique choice of the lifts $P^{\mathrm{gr}}(v \bullet 0)$, $L^{\mathrm{gr}}(v \bullet 0)$ $(v \in W^0)$ such that, if $\mathcal{P}_v^{\mathrm{gr}}$, resp. $\mathcal{L}_v^{\mathrm{gr}}$ is the object of $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$, resp. $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$, such that $P^{\mathrm{gr}}(v \bullet 0) \cong \widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{P}_v^{\mathrm{gr}})$, resp. $L^{\mathrm{gr}}(v \bullet 0) \cong \widetilde{\epsilon}_0^{\mathcal{B}}(\mathcal{L}_v^{\mathrm{gr}})$, for all $w \in W^0$ we have in the category $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$:

$$\kappa_{\mathcal{B}}^{-1} \mathcal{P}^{\mathrm{gr}}_{\tau_0 w} \;\cong\; \zeta(\mathcal{L}_w^{\mathrm{gr}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho). \tag{8.1.1}$$

Let us remark that the functors $\widetilde{\gamma}_0^{\mathcal{B}}$ (of Theorem 6.3.4), $\widetilde{\epsilon}_0^{\mathcal{B}}$ (of Theorem 7.2.1) and $\kappa_{\mathcal{B}}$ commute with the shifts in both the cohomological and the internal grading, by definition. The functor ζ (of Lemma 4.2.1) commutes with the shift in the cohomological grading, but not in the internal one. More precisely, for $\mathcal{F}$ in the category $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$ and $j \in \mathbb{Z}$ one has $\zeta(\mathcal{F}\langle j \rangle) = \zeta(\mathcal{F})[j]\langle j \rangle$. The unicity in Theorem 4.4.3 follows easily from these remarks, using the fact that each lift $P^{\mathrm{gr}}(v \bullet 0)$ and $L^{\mathrm{gr}}(v \bullet 0)$ $(v \in W^0)$ is defined up to a shift $\langle j \rangle$ $(j \in \mathbb{Z})$.

The proof of the existence statement in Theorem 4.4.3 will occupy the rest of this section.

8.2 Koszul dual of the reflection functors

Our proof of statement (‡) (hence also of Theorem 4.4.3) is based on the following result, which shows that the reflection functor $\mathfrak{R}_\delta^{\mathrm{gr}}$ $(\delta \in \Phi_{\mathrm{aff}})$ is conjugate to the semi-simple functor $\mathfrak{S}_\delta^{\mathrm{gr}}$ under Koszul duality, up to some shifts.

Theorem 8.2.1. *For all $\delta \in \Phi_{\mathrm{aff}}$ we have an isomorphism of endofunctors of the category $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$:*

$$(\kappa_{\mathcal{B}})^{-1} \circ \mathfrak{R}_\delta^{\mathrm{gr}} \circ \kappa_{\mathcal{B}} \;\cong\; \mathrm{Shift}_{-\rho} \circ \mathfrak{S}_\delta^{\mathrm{gr}} \circ \mathrm{Shift}_\rho \; [1]\langle 2 \rangle.$$

Proof. By definition of the functors $\mathfrak{R}_{\alpha_0}^{\mathrm{gr}}$ (see equation (6.2.1)) and $\mathfrak{S}_{\alpha_0}^{\mathrm{gr}}$ (see equation (7.3.5)), it is enough to prove the isomorphism for $\delta \in \Phi$. From now on we write α instead of δ. We will derive the theorem from the general results of subsections 2.4 and 2.5.

First, consider the inclusion of vector bundles

$$j_\alpha : (\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)} \hookrightarrow \widetilde{\mathcal{N}}^{(1)}.$$

We apply to this inclusion the constructions of 2.5, with $X = \mathcal{B}^{(1)}$, $E = (\mathfrak{g}^* \times \mathcal{B})^{(1)} \cong E^*$,

$F_1 = (\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)}$, $F_2 = \widetilde{\mathcal{N}}^{(1)}$. Then we have

$$\begin{gathered} F_1^\perp = (\widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)}, \quad F_2^\perp = \widetilde{\mathfrak{g}}^{(1)}, \\ n_1 = \mathrm{rk}(F_1) = \dim(\mathfrak{g}/\mathfrak{b}) - 1, \quad n_2 = \mathrm{rk}(F_2) = \dim(\mathfrak{g}/\mathfrak{b}), \\ \mathcal{L}_1 = \Lambda^{n_1}(\mathcal{F}_1) = \mathcal{O}_{\mathcal{B}^{(1)}}(-2\rho + \alpha), \quad \mathcal{L}_2 = \Lambda^{n_2}(\mathcal{F}_2) = \mathcal{O}_{\mathcal{B}^{(1)}}(-2\rho). \end{gathered}$$

We denote by

$$\widehat{\pi_{\alpha,1}} : \big(\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B}\big)^{(1)} \to \big((\widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B}) \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B}\big)^{(1)}$$

the morphism of dg-schemes induced by the inclusion $\widetilde{\mathfrak{g}}^{(1)} \hookrightarrow (\widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)}$. We also denote by

$$\begin{aligned} \kappa_{\mathcal{B}} : \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) &\xrightarrow{\sim} \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}), \\ \kappa^\alpha : \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)}) &\xrightarrow{\sim} \mathrm{DGCoh}^{\mathrm{gr}}(((\widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B}) \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \end{aligned}$$

the Koszul duality equivalences (see Theorem 2.3.11). Consider the diagram:

$$\begin{tikzcd} \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, shift left, "L(\widetilde{j_{\alpha \mathbb{G}_\mathbf{m}}})^*"] \arrow[d, "\kappa_{\mathcal{B}}", "\wr"'] & \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)}) \arrow[l, shift left, "R(\widetilde{j_{\alpha \mathbb{G}_\mathbf{m}}})_*"] \arrow[d, "\kappa^\alpha", "\wr"'] \\ \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[r, shift left, "R(\widehat{\pi_{\alpha,1}}_{\mathbb{G}_\mathbf{m}})_*"] & \mathrm{DGCoh}^{\mathrm{gr}}(((\widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B}) \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[l, shift left, "L(\widehat{\pi_{\alpha,1}}_{\mathbb{G}_\mathbf{m}})^*"] \end{tikzcd}$$

where the functors are defined as in 2.5. Applying Proposition 2.5.4, one obtains isomorphisms of functors

$$\left\{ \begin{array}{lcl} \kappa^\alpha \circ L(\widetilde{j_{\alpha \mathbb{G}_\mathbf{m}}})^* & \cong & R(\widehat{\pi_{\alpha,1}}_{\mathbb{G}_\mathbf{m}})_* \circ \kappa_{\mathcal{B}}, \\ \kappa_{\mathcal{B}} \circ R(\widetilde{j_{\alpha \mathbb{G}_\mathbf{m}}})_* & \cong & \big(L(\widehat{\pi_{\alpha,1}}_{\mathbb{G}_\mathbf{m}})^* \circ \kappa^\alpha\big) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}}(\alpha)[-1]\langle -2 \rangle. \end{array} \right. \tag{8.2.2}$$

Now, consider the base change

$$\rho_\alpha : (\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)} \to \widetilde{\mathcal{N}}_\alpha^{(1)}.$$

We apply the constructions of 2.4 to this base change, with $X = \mathcal{B}^{(1)}$, $Y = (\mathcal{P}_\alpha)^{(1)}$, $E = (\mathfrak{g}^* \times \mathcal{P}_\alpha)^{(1)}$, $F = \widetilde{\mathcal{N}}_\alpha^{(1)}$. We denote by

$$\widehat{\pi_{\alpha,2}} : \big((\widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B}) \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B}\big)^{(1)} \to \big(\widetilde{\mathfrak{g}}_\alpha \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}_\alpha} \mathcal{P}_\alpha\big)^{(1)}$$

the morphism of dg-schemes induced by the base change $\widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B} \to \widetilde{\mathfrak{g}}_\alpha$. We have the Koszul duality equivalences

$$\kappa^\alpha : \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)}) \xrightarrow{\sim} \mathrm{DGCoh}^{\mathrm{gr}}(((\widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B}) \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}),$$

already used above, and

$$\kappa_\alpha : \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}_\alpha^{(1)}) \xrightarrow{\sim} \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}}_\alpha \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}_\alpha} \mathcal{P}_\alpha)^{(1)}).$$

Consider the diagram

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathcal{N}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B})^{(1)}) \arrow[r, shift left, "R(\widetilde{\rho_\alpha}_{\mathbb{G}_{\mathbf{m}}})_*"] \arrow[d, "\kappa^\alpha", "\wr"'] & \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}_\alpha^{(1)}) \arrow[l, shift left, "L(\widetilde{\rho_\alpha}_{\mathbb{G}_{\mathbf{m}}})^*"] \arrow[d, "\kappa_\alpha", "\wr"'] \\
\mathrm{DGCoh}^{\mathrm{gr}}(((\widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B}) \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \arrow[r, shift left, "R(\widehat{\pi_{\alpha,2}}_{\mathbb{G}_{\mathbf{m}}})_*"] & \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}}_\alpha \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}_\alpha} \mathcal{P}_\alpha)^{(1)}) \arrow[l, shift left, "L(\widehat{\pi_{\alpha,2}}_{\mathbb{G}_{\mathbf{m}}})^*"]
\end{tikzcd}$$

where the functors are defined as in 2.4. Applying Proposition 2.4.6, one obtains isomorphisms of functors

$$\begin{cases} R(\widehat{\pi_{\alpha,2}}_{\mathbb{G}_{\mathbf{m}}})_* \circ \kappa^\alpha & \cong & \kappa_\alpha \circ R(\widetilde{\rho_\alpha}_{\mathbb{G}_{\mathbf{m}}})_*, \\ \kappa^\alpha \circ L(\widetilde{\rho_\alpha}_{\mathbb{G}_{\mathbf{m}}})^* & \cong & L(\widehat{\pi_{\alpha,2}}_{\mathbb{G}_{\mathbf{m}}})^* \circ \kappa_\alpha. \end{cases} \tag{8.2.3}$$

Consider the morphism $\widehat{\pi}_\alpha$. The composition $\widetilde{\mathfrak{g}} \hookrightarrow \widetilde{\mathfrak{g}}_\alpha \times_{\mathcal{P}_\alpha} \mathcal{B} \twoheadrightarrow \widetilde{\mathfrak{g}}_\alpha$ coincides with the morphism $\widetilde{\pi}_\alpha$. Hence we have $\widehat{\pi}_\alpha = \widehat{\pi_{\alpha,2}} \circ \widehat{\pi_{\alpha,1}}$. It follows that $R(\widehat{\pi}_{\alpha,\mathbb{G}_{\mathbf{m}}})_* \cong R(\widehat{\pi_{\alpha,2}}_{\mathbb{G}_{\mathbf{m}}})_* \circ R(\widehat{\pi_{\alpha,1}}_{\mathbb{G}_{\mathbf{m}}})_*$ and $L(\widehat{\pi}_{\alpha,\mathbb{G}_{\mathbf{m}}})^* \cong L(\widehat{\pi_{\alpha,1}}_{\mathbb{G}_{\mathbf{m}}})^* \circ L(\widehat{\pi_{\alpha,2}}_{\mathbb{G}_{\mathbf{m}}})^*$ (see isomorphisms (1.7.7) and (1.7.8)). Hence formulas (8.2.2) and (8.2.3) allow us to compute $(\kappa_{\mathcal{B}})^{-1} \circ \mathfrak{R}_\alpha^{\mathrm{gr}} \circ \kappa_{\mathcal{B}} = (\kappa_{\mathcal{B}})^{-1} \circ L(\widehat{\pi}_{\alpha,\mathbb{G}_{\mathbf{m}}})^* \circ R(\widehat{\pi}_{\alpha,\mathbb{G}_{\mathbf{m}}})_* \circ \kappa_{\mathcal{B}}$. Namely, we obtain isomorphisms

$$R(\widehat{\pi_\alpha}_{\mathbb{G}_{\mathbf{m}}})_* \circ \kappa_{\mathcal{B}} \cong \kappa_\alpha \circ R(\widetilde{\rho_\alpha}_{\mathbb{G}_{\mathbf{m}}})_* \circ L(\widetilde{j_\alpha}_{\mathbb{G}_{\mathbf{m}}})^*$$

and

$$(\kappa_{\mathcal{B}})^{-1} \circ L(\widehat{\pi_\alpha}_{\mathbb{G}_{\mathbf{m}}})^* \cong \big(R(\widetilde{j_\alpha}_{\mathbb{G}_{\mathbf{m}}})_* \circ L(\widetilde{\rho_\alpha}_{\mathbb{G}_{\mathbf{m}}})^* \circ (\kappa_\alpha)^{-1}\big) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}}(-\alpha)[1]\langle 2 \rangle.$$

Hence, finally,

$$(\kappa_{\mathcal{B}})^{-1} \circ \mathfrak{R}_\alpha^{\mathrm{gr}} \circ \kappa_{\mathcal{B}} \cong \\ \big(R(\widetilde{j_\alpha}_{\mathbb{G}_{\mathbf{m}}})_* \circ L(\widetilde{\rho_\alpha}_{\mathbb{G}_{\mathbf{m}}})^* \circ R(\widetilde{\rho_\alpha}_{\mathbb{G}_{\mathbf{m}}})_* \circ L(\widetilde{j_\alpha}_{\mathbb{G}_{\mathbf{m}}})^*\big) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}}(-\alpha)[1]\langle 2 \rangle.$$

Comparing this with the definition of $\mathfrak{S}_\alpha^{\mathrm{gr}}$ in subsection 7.3, one obtains the result. □

8.3 Action of the braid group on $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$

Recall that we have defined in subsection 5.2, respectively 7.3, an action of the group B'_{aff} on the category $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}^{(1)})$, respectively $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$. Consider the diagram:

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, "\mathbf{K}_b^{\mathrm{gr}}"] \arrow[d, "\eta"] & \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[d, "\eta"] \\
\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, "\mathbf{K}_b^{\mathbb{G}_{\mathbf{m}}}"] & \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}^{(1)})
\end{tikzcd}$$

where η is the functor defined in subsection 4.2 (see also equation (2.3.6)).

Lemma 8.3.1. *For any $\mathcal{M} \in \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$, there exists an isomorphism*[11]

$$\eta \circ \mathbf{K}_b^{\mathrm{gr}}(\mathcal{M}) \cong \mathbf{K}_b^{\mathbb{G}_{\mathbf{m}}} \circ \eta(\mathcal{M}).$$

Proof. It is sufficient to prove the isomorphism on a set of generators of B'_{aff}. For $b = \theta_x$ ($x \in \mathbb{X}$), the result follows from the fact that the Koszul duality $\kappa_{\mathcal{B}}$ commutes with the twist by a line bundle on $\mathcal{B}^{(1)}$. Hence we only have to prove it for $b = T_\alpha$ for $\alpha \in \Phi$. Let us fix such an α. Recall the distinguished triangle of functors of Lemma 6.2.4. It induces a triangle

$$\mathrm{Id}\langle 1\rangle \to \mathrm{Shift}_\rho \circ (\kappa_{\mathcal{B}})^{-1} \circ \mathfrak{R}_\alpha^{\mathrm{gr}} \circ \kappa_{\mathcal{B}} \circ \mathrm{Shift}_{-\rho}\langle -1\rangle \to \mathbf{K}_{T_\alpha}^{\mathrm{gr}}.$$

Using the isomorphism provided by Theorem 8.2.1, we obtain a triangle

$$\mathrm{Id}\langle 1\rangle \to \mathfrak{S}_\alpha^{\mathrm{gr}}[1]\langle 1\rangle \to \mathbf{K}_{T_\alpha}^{\mathrm{gr}}.$$

For any $\mathcal{M}$ in $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$ we thus have a distinguished triangle

$$\eta(\mathcal{M})[-1]\langle 1\rangle \to \eta \circ \mathfrak{S}_\alpha^{\mathrm{gr}}(\mathcal{M})\langle 1\rangle \to \eta \circ \mathbf{K}_{T_\alpha}^{\mathrm{gr}}(\mathcal{M}) \tag{8.3.2}$$

(observe that $\eta(\mathcal{F}\langle j\rangle) = \eta(\mathcal{F})[-j]\langle j\rangle$). By diagram (7.3.3) we have $\eta \circ \mathfrak{S}_\alpha^{\mathrm{gr}} = \mathfrak{S}_\alpha^{\mathbb{G}_{\mathbf{m}}} \circ \eta$.

Now the exact sequence of $\mathbb{G}_{\mathbf{m}}$-equivariant sheaves (5.3.4) induces a distinguished triangle of functors

$$\mathfrak{S}_\alpha^{\mathbb{G}_{\mathbf{m}}}\langle 1\rangle \to \mathbf{K}_{T_\alpha}^{\mathbb{G}_{\mathbf{m}}} \to \mathrm{Id}\langle 1\rangle. \tag{8.3.3}$$

Identifying triangle (8.3.2) with triangle (8.3.3) applied to $\eta(\mathcal{M})$, one obtains the isomorphisms for $b = T_\alpha$. □

Remark 8.3.4. It follows in particular from this lemma that the diagrams (7.3.3) and (7.3.4), with α replaced by α_0, are commutative on objects, *i.e.* for any $\mathcal{M}$ in $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})$ there exists an isomorphism $\eta \circ \mathfrak{S}_{\alpha_0}^{\mathrm{gr}}(\mathcal{M}) \cong \mathfrak{S}_{\alpha_0}^{\mathbb{G}_{\mathbf{m}}} \circ \eta(\mathcal{M})$, and similarly for the second diagram.

8.4 End of the proof of Theorem 4.4.3

In this subsection we finally give a proof of the existence statement in (‡) (see 8.1), by induction on $\ell(w)$.

To begin induction, let us consider some $w \in W^0$ with $\ell(w) = 0$. Write $w = v \cdot t_\mu$. We have seen in Corollary 7.2.2 that $\mathcal{L}_w \cong j_* \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho + \mu)[\ell(v)]$. Let us set

$$\mathcal{L}_w^{\mathrm{gr}} := j_* \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho + \mu)[\ell(v)]\langle N - \ell(v)\rangle,$$

where $N = \# R^+$, and $j_* \mathcal{O}_{\mathcal{B}^{(1)}}$ is endowed with its natural (trivial) $\mathbb{G}_{\mathbf{m}}$-equivariant structure. It is clear that $L^{\mathrm{gr}}(w \bullet 0) := \widetilde{\epsilon}_0^{\mathcal{B}}(\mathcal{L}_w^{\mathrm{gr}})$ is a lift of $L(w \bullet 0)$ as a graded module (see the proof of Proposition 7.2.3). As in subsection 3.1 we denote by $\mathcal{T}_{\mathcal{B}^{(1)}}$ the tangent sheaf of

[11] It is not clear from our proof whether or not these isomorphisms yield an isomorphism of functors. This is not important for our arguments, hence we will not consider this issue.

$\mathcal{B}^{(1)}$. By definition of Koszul duality (see equation (3.1.2)) and the remarks on shifts at the end of subsection 8.1 we have

$$\begin{aligned}\kappa_{\mathcal{B}}\big(\zeta(\mathcal{L}_w^{\mathrm{gr}}) \otimes \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho)\big) &\cong \kappa_{\mathcal{B}}\big(\mathcal{O}_{\mathcal{B}^{(1)}}(-2\rho+\mu)[N]\langle N-\ell(v)\rangle\big) \\ &\cong \Lambda(\mathcal{T}^{\vee}_{\mathcal{B}^{(1)}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}}(\mu)\langle -N-\ell(v)\rangle.\end{aligned}$$

We set

$$\mathcal{P}^{\mathrm{gr}}_{\tau_0 w} := \Lambda(\mathcal{T}^{\vee}_{\mathcal{B}^{(1)}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}}(\mu)\langle -N-\ell(v)\rangle.$$

It follows from (6.4.6) that $P^{\mathrm{gr}}(\tau_0 w \bullet 0) := \widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{P}^{\mathrm{gr}}_{\tau_0 w})$ is a lift of $P(\tau_0 w \bullet 0)$ as a graded module (see also the proof of Proposition 6.5.1). Moreover, isomorphism (8.1.1) is true by definition. This concludes the proof in the case $\ell(w)=0$.

Now, consider some $w \in W^0$, and assume the result is known for all $v \in W^0$ with $\ell(v) < \ell(w)$. For all such v, the lifts $L^{\mathrm{gr}}(v \bullet 0)$ of $L(v \bullet 0)$ and $P^{\mathrm{gr}}(\tau_0 v \bullet 0)$ of $P(\tau_0 v \bullet 0)$ are fixed such that, if $\mathcal{L}_v^{\mathrm{gr}}$, respectively $\mathcal{P}^{\mathrm{gr}}_{\tau_0 v}$ is the object (unique up to isomorphism) of $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$, respectively $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$, such that $\widetilde{\epsilon}_0^{\mathcal{B}}(\mathcal{L}_v^{\mathrm{gr}}) = L^{\mathrm{gr}}(v \bullet 0)$, respectively $\widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{P}^{\mathrm{gr}}_{\tau_0 v}) \cong P^{\mathrm{gr}}(\tau_0 v \bullet 0)$, isomorphism (8.1.1) is satisfied. Choose some $\delta \in \Phi_{\mathrm{aff}}$ such that, for $s = s_\delta$, one has $ws \in W^0$ and $ws \bullet 0 < w \bullet 0$, *i.e.* $\ell(ws) < \ell(w)$. In particular we have

$$\kappa_{\mathcal{B}}\big(\zeta(\mathcal{L}^{\mathrm{gr}}_{ws}) \otimes \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho)\big) \cong \mathcal{P}^{\mathrm{gr}}_{\tau_0 ws}.$$

Applying $\mathfrak{R}^{\mathrm{gr}}_\delta$ and using Theorem 8.2.1, it follows that

$$\kappa_{\mathcal{B}}(\mathfrak{S}^{\mathrm{gr}}_\delta \circ \zeta(\mathcal{L}^{\mathrm{gr}}_{ws}) \otimes \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho))[1]\langle 1\rangle \cong \mathfrak{R}^{\mathrm{gr}}_\delta \mathcal{P}^{\mathrm{gr}}_{\tau_0 ws}\langle -1\rangle. \tag{8.4.1}$$

As in the proof of Proposition 6.5.1, the image under the forgetful functor For : $\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})_0) \to \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$ of $\widetilde{\gamma}_0^{\mathcal{B}}(\mathfrak{R}^{\mathrm{gr}}_\delta \mathcal{P}^{\mathrm{gr}}_{\tau_0 ws})$ is $R_\delta P(\tau_0 ws \bullet 0)$. Hence there exists a lift $P^{\mathrm{gr}}(\tau_0 w \bullet 0)$ of $P(\tau_0 w \bullet 0)$, and graded finite dimensional vector spaces $V_{\tau_0 v}$ ($v \in W^0$, $\ell(v) < \ell(w)$) such that

$$\widetilde{\gamma}_0^{\mathcal{B}}(\mathfrak{R}^{\mathrm{gr}}_\delta \mathcal{P}^{\mathrm{gr}}_{\tau_0 ws})\langle -1\rangle \cong P^{\mathrm{gr}}(\tau_0 w \bullet 0) \oplus \big(\bigoplus_{\substack{v \in W^0 \\ \ell(v)<\ell(w)}} P^{\mathrm{gr}}(\tau_0 v \bullet 0) \otimes_{\Bbbk} V_{\tau_0 v}\big) \tag{8.4.2}$$

(see again the proof of Proposition 6.5.1).

Now let us consider the left hand side of equation (8.4.1). By diagram (7.3.4) and Remark 8.3.4 we have $\mathfrak{S}^{\mathrm{gr}}_\delta \circ \zeta(\mathcal{L}^{\mathrm{gr}}_{ws}) \cong \zeta \circ \mathfrak{S}^{\mathbb{G}_{\mathbf{m}}}_\delta(\mathcal{L}^{\mathrm{gr}}_{ws})$. As in the proof of Proposition 7.2.3, the image of $\widetilde{\epsilon}_0^{\mathcal{B}}(\mathfrak{S}^{\mathbb{G}_{\mathbf{m}}}_\delta \mathcal{L}^{\mathrm{gr}}_{ws})$ under the forgetful functor For : $\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})^0) \to \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0)$ is the module $Q_\delta(ws)$. Hence, again as in the proof of Proposition 7.2.3, there is a lift $L^{\mathrm{gr}}(w \bullet 0)$ of $L(w \bullet 0)$ as a graded module, an object $\mathcal{Q}^{\mathrm{gr}}$ of $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$, and an isomorphism

$$\widetilde{\epsilon}_0^{\mathcal{B}}(\mathfrak{S}^{\mathbb{G}_{\mathbf{m}}}_\delta \mathcal{L}^{\mathrm{gr}}_{ws}) \cong L^{\mathrm{gr}}(w \bullet 0)\langle -1\rangle \oplus \widetilde{\epsilon}_0^{\mathcal{B}}(\mathcal{Q}^{\mathrm{gr}}).$$

Let $\mathcal{L}^{\mathrm{gr}}_w$ be the object of $\mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_{\mathbf{m}}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$ such that $\widetilde{\epsilon}_0^{\mathcal{B}}(\mathcal{L}^{\mathrm{gr}}_w) = L^{\mathrm{gr}}(w \bullet 0)$. Then, as $\widetilde{\epsilon}_0^{\mathcal{B}}$ is fully faithful, $\mathcal{L}^{\mathrm{gr}}_w$ is a direct factor of $\mathfrak{S}^{\mathbb{G}_{\mathbf{m}}}_\delta \mathcal{L}^{\mathrm{gr}}_{ws}\langle 1\rangle$. Hence, using the remarks on the shifts at the end of subsection 8.1, $\kappa_{\mathcal{B}}(\zeta(\mathcal{L}_w^{\mathrm{gr}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho))$ is a direct factor of the left hand side of equation (8.4.1), thus also of its right hand right.

Let us define

$$\mathcal{P}_{\tau_0 w}^{\mathrm{gr}} := \kappa_{\mathcal{B}}(\zeta(\mathcal{L}_w^{\mathrm{gr}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho)). \tag{8.4.3}$$

To conclude the proof of the induction step, it is enough to prove that

$$\widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{P}_{\tau_0 w}^{\mathrm{gr}}) \;\cong\; P^{\mathrm{gr}}(\tau_0 w \bullet 0). \tag{8.4.4}$$

By definition, $\mathcal{P}_{\tau_0 w}^{\mathrm{gr}}$ is a direct factor of the object appearing in equation (8.4.1). Hence $\widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{P}_{\tau_0 w}^{\mathrm{gr}})$ is a direct factor of the object appearing in (8.4.2). In particular, it is concentrated in cohomological degree 0, *i.e.* it is a graded $(\mathcal{U}\mathfrak{g})_0^{\hat{0}}$-module. Let us show that it is indecomposable. By Proposition 5.6.2(i), it is enough to show that its endomorphism algebra is local. This algebra is isomorphic to

$$\begin{aligned}
\mathrm{End}_{\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})_0)}(\widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{P}_{\tau_0 w}^{\mathrm{gr}})) \;&\cong\; \mathrm{End}_{\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})}(\mathcal{P}_{\tau_0 w}^{\mathrm{gr}}) \\
&\cong\; \mathrm{End}_{\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}^{(1)})}(\mathcal{L}_w^{\mathrm{gr}}) \\
&\cong\; \mathrm{End}_{\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})^0)}(L^{\mathrm{gr}}(w \bullet 0)) \\
&\cong\; \Bbbk
\end{aligned}$$

Here the first isomorphism follows from the fact that $\widetilde{\gamma}_0^{\mathcal{B}}$ is fully faithful. The second one follows from definition (8.4.3), and the fact that $\kappa_{\mathcal{B}}$ and ζ are fully faithful. The third isomorphism follows from the definition of $\mathcal{L}_w^{\mathrm{gr}}$ and the fact that $\widetilde{\epsilon}_0^{\mathcal{B}}$ is fully faithful. It follows that $\widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{P}_{\tau_0 w}^{\mathrm{gr}})$ is an indecomposable graded $(\mathcal{U}\mathfrak{g})_0^{\hat{0}}$-module.

By the Krull-Schmidt theorem (see Proposition 5.6.2(ii)), we deduce that $\widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{P}_{\tau_0 w}^{\mathrm{gr}})$ is one of the indecomposable summands appearing in the right hand side of (8.4.2). Hence, to conclude the proof of (8.4.4) it is enough to prove that there cannot exist some $i \in \mathbb{Z}$ and some $v \in W^0$ with $\ell(v) < \ell(w)$ such that

$$\widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{P}_{\tau_0 w}^{\mathrm{gr}}) \;\cong\; P^{\mathrm{gr}}(\tau_0 v \bullet 0)\langle i \rangle.$$

Let us assume that there exist such an i and such a v. By induction hypothesis we have $P^{\mathrm{gr}}(\tau_0 v \bullet 0)\langle i \rangle \cong \widetilde{\gamma}_0^{\mathcal{B}}(\mathcal{P}_{\tau_0 v}^{\mathrm{gr}}\langle i \rangle)$, and

$$\mathcal{P}_{\tau_0 v}^{\mathrm{gr}}\langle i \rangle \;\cong\; \kappa_{\mathcal{B}}(\zeta(\mathcal{L}_v^{\mathrm{gr}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho))\langle i \rangle.$$

Hence, as $\widetilde{\gamma}_0^{\mathcal{B}}$, $\kappa_{\mathcal{B}}$ and ζ are fully faithful, by definition (8.4.3) we have

$$\mathcal{L}_w^{\mathrm{gr}} \;\cong\; \mathcal{L}_v^{\mathrm{gr}}[-i]\langle i \rangle.$$

Applying $\widetilde{\epsilon}_0^{\mathcal{B}}$ one obtains

$$L^{\mathrm{gr}}(w \bullet 0) \;\cong\; L^{\mathrm{gr}}(v \bullet 0)[-i]\langle i \rangle,$$

which is a contradiction as $v \neq w$.

This concludes the proof of (‡), hence also of Theorem 4.4.3.

8.5 Remark on other alcoves

In Theorem 4.4.3, the objects $\mathcal{L}_w$, respectively $\mathcal{P}_w$, correspond to simple, respectively projective, modules for any choice of a weight $\lambda \in C_0$, *i.e.* they are the simple, respectively projective, objects for the t-structure on the category $\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$, respectively $\mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$, assigned to the fundamental alcove (see [Bez06b, 2.1.5] for details on this point of view). We could also consider the simple, respectively projective, objects for the t-structure assigned to another alcove C_1, *i.e.* the objects which are sent by the equivalence $\epsilon_\lambda^{\mathcal{B}}$, respectively $\widehat{\gamma}_\lambda^{\mathcal{B}}$, to the simple, respectively projective, modules, for any $\lambda \in C_1 \cap \mathbb{X}$. The different t-structures are related by the braid group action, which commutes with $\kappa_{\mathcal{B}}$ (see Lemma 8.3.1). Hence a statement similar to Theorem 4.4.3 is true for any alcove.

More precisely, let C be the intersection of an alcove with $\mathbb{X}$. Let $y \in W_{\mathrm{aff}}$ be the unique element such that $C = y \bullet C_0$. Then there exist unique objects $\mathcal{L}_w^y \in \mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)})$, $\mathcal{P}_w^y \in \mathrm{DGCoh}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)})$ ($w \in W^0$) such that for any $\lambda \in C$ and $w \in W^0$ we have

$$\begin{cases} \epsilon_\lambda^{\mathcal{B}}(\mathcal{L}_w^y) & \cong & L(w \bullet (y^{-1} \bullet \lambda)) \\ \widehat{\gamma}_\lambda^{\mathcal{B}}(\mathcal{P}_w^y) & \cong & P(w \bullet (y^{-1} \bullet \lambda)) \end{cases} . \tag{8.5.1}$$

(Observe that, in this formula, $y^{-1} \bullet \lambda \in C_0$.) Indeed, there is an element $\overline{y} \in B'_{\mathrm{aff}}$ such that $\gamma_\lambda^{\mathcal{B}} \cong \gamma_{y^{-1}\bullet\lambda}^{\mathcal{B}} \circ \mathbf{J}_{\overline{y}}$ for any $\lambda \in C$ (see [Bez06b] and [BMR06, section 2] for details). Here $\overline{y}$ is not unique, but the functor $\mathbf{J}_{\overline{y}}$ is clearly unique (up to isomorphism). Then, if we set $\mathcal{L}_w^y := \mathbf{K}_{\overline{y}}^{-1}(\mathcal{L}_w)$ and $\mathcal{P}_w^y := (\mathbf{J}_{\overline{y}}^{\mathrm{dg}})^{-1}(\mathcal{P}_w)$, one easily checks that isomorphisms (8.5.1) are satisfied.

Also, if we define $\mathcal{L}_w^{y,\mathrm{gr}} := (\mathbf{K}_{\overline{y}}^{\mathbb{G}_\mathbf{m}})^{-1}(\mathcal{L}_w^{\mathrm{gr}})$ and $\mathcal{P}_w^{y,\mathrm{gr}} := (\mathbf{J}_{\overline{y}}^{\mathrm{dg,gr}})^{-1}(\mathcal{P}_w^{\mathrm{gr}})$, these objects are lifts of the $\mathcal{L}_w^y$'s and $\mathcal{P}_w^y$'s, and we have isomorphisms $\kappa_{\mathcal{B}}^{-1}\mathcal{P}_{\tau_0 w}^{y,\mathrm{gr}} \cong \zeta(\mathcal{L}_w^{y,\mathrm{gr}}) \otimes_{\mathcal{O}_{\mathcal{B}^{(1)}}} \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho)$ for all $w \in W^0$. (The isomorphisms follow from the fact that $\kappa_{\mathcal{B}}$ and ζ commute with the braid group action, see Lemma 8.3.1.)

Similarly, for any $\lambda \in C$ there are "graded versions" of the functors $\epsilon_\lambda^{\mathcal{B}}$, $\widehat{\gamma}_\lambda^{\mathcal{B}}$, with properties similar to those of $\widetilde{\epsilon}_0^{\mathcal{B}}$, $\widetilde{\gamma}_0^{\mathcal{B}}$, and statements similar to statement (‡) of subsection 8.1.

9 Application to Koszulity of the regular blocks of $(\mathcal{U}\mathfrak{g})_0$

In this section we derive from Theorem 4.4.3 (or rather from the equivalent statement (‡) of 8.1) that, for $\lambda \in C_0$, the category $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^\lambda)$ is "controlled" by a Koszul ring, whose Koszul dual controls the category $\mathrm{Mod}_\lambda^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$. These results can be considered as counterparts in positive characteristic of the results in [Soe90] and [BGS96]. They also extend some results of [AJS94, section 18].

We deduce this property from a general criterion for a graded ring to be Morita equivalent to a Koszul ring, proved in 9.2.

9.1 More results on graded algebras

Let A by a $\mathbb{Z}$-graded ring. Recall the notation of 5.6. Following [NvO82, A.I.7], if M is in $\mathrm{Mod}^{\mathrm{gr}}(A)$, we define the *graded radical* $\mathrm{rad}^{\mathrm{gr}}(M)$ of M to be the intersection of all maximal *graded* submodules of M. With this definition, $\mathrm{rad}^{\mathrm{gr}}$ has all the usual properties of the radical (see [NvO82, A.I.7.4]). In particular, if A is considered as an A-module via left multiplication, $\mathrm{rad}^{\mathrm{gr}}(A)$ is a graded two-sided ideal of A, and

$$\mathrm{rad}^{\mathrm{gr}}(A) = \bigcap_{\substack{X \text{ simple} \\ \text{graded } A\text{-module}}} \mathrm{Ann}(X). \tag{9.1.1}$$

From now on in this section we restrict to the following case. Let V be a graded finite dimensional $\Bbbk$-vector space, concentrated in positive degrees. Let $\mathrm{S}(V)$ be the symmetric algebra of V. It is naturally a graded $\Bbbk$-algebra, concentrated in non-negative degrees. We assume that A is a graded $\mathrm{S}(V)$-algebra, which is finitely generated as a $\mathrm{S}(V)$-module. Note in particular that the grading of A is bounded below.

Let us define the finite dimensional graded $\Bbbk$-algebra $\overline{A} := A/(V \cdot A)$. By Theorem 5.6.1(ii) and Corollary 5.6.4(i), the simple $\overline{A}$-modules are exactly the images of the simple graded $\overline{A}$-modules under the forgetful functor. Comparing (9.1.1) with [CR81, 5.5], we deduce that

$$\mathrm{rad}(\overline{A}) = \mathrm{rad}^{\mathrm{gr}}(\overline{A}). \tag{9.1.2}$$

A proof entirely similar to that of [CR81, 5.22] yields the following result.

Proposition 9.1.3. (i) *The morphism $A \to \overline{A}$ induces an isomorphism of graded rings $A/\mathrm{rad}^{\mathrm{gr}}(A) \cong \overline{A}/\mathrm{rad}^{\mathrm{gr}}(\overline{A})$.*

(ii) *For $k \gg 0$, $(\mathrm{rad}^{\mathrm{gr}}(A))^k \subseteq V \cdot A$.*

We denote by $\mathrm{Hom}_{A,\mathbb{Z}}(M,N)$ the morphisms in the abelian category $\mathrm{Mod}^{\mathrm{gr}}(A)$, and by $\mathrm{Ext}^i_{A,\mathbb{Z}}(M,N)$ the corresponding extension groups. By [AJS94, E.6] we also have:

Lemma 9.1.4. (i) *Let $M \in \mathrm{Mod}^{\mathrm{fg,gr}}(A)$. If M is indecomposable in $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$, then $\mathrm{End}_{A,\mathbb{Z}}(M)$ is a local algebra.*

(ii) *The Krull-Schmidt theorem holds in $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$.*

If L is a simple graded A-module, then $V \cdot L = 0$. Indeed, $V \cdot L$ is a graded submodule of L and, as L is bounded below and V is in positive degrees, we cannot have $L = V \cdot L$. Hence the simple graded A-modules are the simple graded $\overline{A}$-modules.

Let $L_1, \ldots, L_n$ be representatives of the simple non-graded $\overline{A}$-modules, and, for $i = 1 \ldots r$, let L_i^{gr} be a lift of L_i as a graded $\overline{A}$-module (it exists by Theorem 5.6.1(ii)). Using Corollary 5.6.4(i) and Theorem 5.6.1(iv), the $L_i\langle j \rangle$ are representatives of the simple graded $\overline{A}$-modules, hence also of the simple graded A-modules. As the ring $\overline{A}/\mathrm{rad}(\overline{A})$ is semi-simple (see *e.g.* [CR81, 5.19]), using (9.1.2), Proposition 9.1.3(i) and Corollary 5.6.4(ii), every graded $A/\mathrm{rad}^{\mathrm{gr}}(A)$-module is semi-simple in $\mathrm{Mod}^{\mathrm{fg,gr}}(A/\mathrm{rad}^{\mathrm{gr}}(A))$. Using also Lemma 9.1.4, every object of $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$ has a projective cover. For $i = 1 \dots r$, let P_i^{gr} be a projective cover of L_i^{gr}. We have

$$L_i^{\mathrm{gr}} = P_i^{\mathrm{gr}} / \mathrm{rad}^{\mathrm{gr}}(P_i^{\mathrm{gr}}). \tag{9.1.5}$$

We will finally need the following result. For M in $\mathrm{Mod}^{\mathrm{gr}}(A)$ and $i \geq 0$, we define $\mathrm{rad}^{\mathrm{gr},i}(M)$ by induction, setting $\mathrm{rad}^{\mathrm{gr},0}(M) = M$, and $\mathrm{rad}^{\mathrm{gr},i}(M) = \mathrm{rad}^{\mathrm{gr}}(\mathrm{rad}^{\mathrm{gr},i-1}(M))$ if $i \geq 1$.

Lemma 9.1.6. *Let M be an object of* $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$.

(i) $\mathrm{rad}^{\mathrm{gr}}(M) = \mathrm{rad}^{\mathrm{gr}}(A) \cdot M$.

(ii) $\bigcap_{i \geq 0} \mathrm{rad}^{\mathrm{gr},i}(M) = \{0\}$.

Proof. The proof of (i) is similar to that of [CR81, 5.29]. As A is noetherian we deduce, by induction on i, that $\mathrm{rad}^{\mathrm{gr},i}(M) = (\mathrm{rad}^{\mathrm{gr}}(A))^i \cdot M$ for $i \geq 0$. By (ii) of Proposition 9.1.3, for $k \gg 0$ we have $(\mathrm{rad}^{\mathrm{gr}}(A))^k \subset V \cdot A$. Hence $\bigcap_{i \geq 0} \mathrm{rad}^{\mathrm{gr},i}(M) \subseteq \bigcap_{i \geq 0} (V^i \cdot M)$. As M is finitely generated over A, it is bounded below. As V is concentrated in positive degrees, we deduce that $\bigcap_{i \geq 0} V^i \cdot M = \{0\}$. This proves (ii). □

9.2 A Koszulity criterion

Recall that a *Koszul ring* $A = \bigoplus_{n \geq 0} A_n$ is a non-negatively graded ring such that A_0 is a semi-simple ring and the graded left A-module $A_0 \cong A/A_{>0}$ admits a graded projective resolution

$$\cdots \to P^2 \to P^1 \to P^0 \to A_0 \to 0$$

such that P^i is generated by its degree i part, for all i. We refer to [BGS96] for generalities on such rings. If A is a Koszul ring, then its *dual Koszul ring* is the graded ring[12]

$$A^! := \big(\bigoplus_{n \geq 0} \mathrm{Ext}_A^n(A_0, A_0)\big)^{\mathrm{op}}$$

(here the Ext-groups are taken in the category of non-graded A-modules). If A_1 is an A_0-module of finite type, then $A^!$ is also a Koszul ring.

If A is a (non graded) ring, one says that A *admits a Koszul grading* if it can be endowed with a grading which makes it a Koszul ring. If A is artinian, this grading is unique (up to automorphism) if it exists (see [BGS96, 2.5.2]).

Theorem 9.2.1. *Let A, L_i, L_i^{gr} be as in subsection* 9.1. *Assume one can choose the lifts L_i^{gr} such that for $i, j = 1, \dots, r$,*

$$\mathrm{Ext}^n_{A,\mathbb{Z}}(L_i^{\mathrm{gr}}, L_j^{\mathrm{gr}}\langle m \rangle) = 0 \quad \textit{unless } n = m. \tag{9.2.2}$$

Then there exists a Koszul ring B which is Morita equivalent to A (as a graded ring).

[12] A Koszul ring is in particular a quadratic ring, and the dual Koszul ring is also the dual quadratic ring. The definition chosen here in easier to state, although it is less concrete.

If $L = \bigoplus_{i=1}^n L_i$, *the ring* $B^!$ *is isomorphic to*

$$\Big(\bigoplus_{n\geq 0} \mathrm{Ext}_A^n(L, L)\Big)^{\mathrm{op}}$$

The proof will occupy the rest of this subsection. Assume that the lifts L_i^{gr} can be chosen so that (9.2.2) is satisfied, and let P_i^{gr} be the projective cover of L_i^{gr}. We begin with the following lemma.

Lemma 9.2.3. *For* $n \geq 0$ *and* $i = 1 \ldots r$,

$$\mathrm{rad}^{\mathrm{gr},n}(P_i^{\mathrm{gr}})/\mathrm{rad}^{\mathrm{gr},n+1}(P_i^{\mathrm{gr}})$$

is a direct sum of simple modules of the form $L_j^{\mathrm{gr}}\langle n\rangle$ $(j \in \{1, \ldots, r\})$.

Proof. We prove the result by induction on $n \geq 0$. It is clear for $n = 0$, by (9.1.5). Let $n \geq 1$, and assume it is true for $n - 1$. The graded A-module $\mathrm{rad}^{\mathrm{gr},n}(P_i^{\mathrm{gr}})/\mathrm{rad}^{\mathrm{gr},n+1}(P_i^{\mathrm{gr}})$ factorizes through an $A/\mathrm{rad}^{\mathrm{gr}}(A)$-module. Using the remarks before (9.1.5) we deduce that it is semi-simple, hence a direct sum of modules $L_j^{\mathrm{gr}}\langle m\rangle$ $(j \in \{1, \ldots, r\}$, $m \in \mathbb{Z})$. The multiplicity of $L_j^{\mathrm{gr}}\langle m\rangle$ is the dimension of $\mathrm{Hom}_{A,\mathbb{Z}}(\mathrm{rad}^{\mathrm{gr},n}(P_i^{\mathrm{gr}})/\mathrm{rad}^{\mathrm{gr},n+1}(P_i^{\mathrm{gr}}), L_j^{\mathrm{gr}}\langle m\rangle)$. By usual properties of $\mathrm{rad}^{\mathrm{gr}}$, we have

$$\mathrm{Hom}_{A,\mathbb{Z}}(\mathrm{rad}^{\mathrm{gr},n}(P_i^{\mathrm{gr}})/\mathrm{rad}^{\mathrm{gr},n+1}(P_i^{\mathrm{gr}}), L_j^{\mathrm{gr}}\langle m\rangle) \;\cong\; \mathrm{Hom}_{A,\mathbb{Z}}(\mathrm{rad}^{\mathrm{gr},n}(P_i^{\mathrm{gr}}), L_j^{\mathrm{gr}}\langle m\rangle).$$

Hence we only have to prove that:

$$\mathrm{Hom}_{A,\mathbb{Z}}(\mathrm{rad}^{\mathrm{gr},n}(P_i^{\mathrm{gr}}), L_j^{\mathrm{gr}}\langle m\rangle) = 0 \quad \text{unless } m = n.$$

Consider the exact sequence

$$0 \to \mathrm{rad}^{\mathrm{gr},n}(P_i^{\mathrm{gr}}) \to \mathrm{rad}^{\mathrm{gr},n-1}(P_i^{\mathrm{gr}}) \to \mathrm{rad}^{\mathrm{gr},n-1}(P_i^{\mathrm{gr}})/\mathrm{rad}^{\mathrm{gr},n}(P_i^{\mathrm{gr}}) \to 0.$$

For $j \in \{1, \ldots, r\}$ and $m \in \mathbb{Z}$, it induces an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{A,\mathbb{Z}}&(\mathrm{rad}^{\mathrm{gr},n-1}(P_i^{\mathrm{gr}})/\mathrm{rad}^{\mathrm{gr},n}(P_i^{\mathrm{gr}}), L_j^{\mathrm{gr}}\langle m\rangle) \\ &\xrightarrow{\lambda} \mathrm{Hom}_{A,\mathbb{Z}}(\mathrm{rad}^{\mathrm{gr},n-1}(P_i^{\mathrm{gr}}), L_j^{\mathrm{gr}}\langle m\rangle) \to \mathrm{Hom}_{A,\mathbb{Z}}(\mathrm{rad}^{\mathrm{gr},n}(P_i^{\mathrm{gr}}), L_j^{\mathrm{gr}}\langle m\rangle) \\ &\qquad\qquad \xrightarrow{\mu} \mathrm{Ext}^1_{A,\mathbb{Z}}(\mathrm{rad}^{\mathrm{gr},n-1}(P_i^{\mathrm{gr}})/\mathrm{rad}^{\mathrm{gr},n}(P_i^{\mathrm{gr}}), L_j^{\mathrm{gr}}\langle m\rangle). \end{aligned}$$

By usual properties of $\mathrm{rad}^{\mathrm{gr}}$, the morphism λ is an isomorphism. Hence μ is injective. Moreover, using induction and property (9.2.2), we have

$$\mathrm{Ext}^1_{A,\mathbb{Z}}(\mathrm{rad}^{\mathrm{gr},n-1}(P_i^{\mathrm{gr}})/\mathrm{rad}^{\mathrm{gr},n}(P_i^{\mathrm{gr}}), L_j^{\mathrm{gr}}\langle m\rangle) = 0 \quad \text{unless } m = n.$$

This finishes the proof. □

We define $P^{\mathrm{gr}} := \bigoplus_{i=1}^{r} P_i^{\mathrm{gr}}$. Let B be the algebra

$$B := \mathrm{Hom}_A(P^{\mathrm{gr}}, P^{\mathrm{gr}})^{\mathrm{op}}.$$

As P^{gr} is finitely generated, B is naturally graded, with n-th component

$$B_n := \mathrm{Hom}_{A,\mathbb{Z}}(P^{\mathrm{gr}}\langle n\rangle, P^{\mathrm{gr}}) \cong \mathrm{Hom}_{A,\mathbb{Z}}(P^{\mathrm{gr}}, P^{\mathrm{gr}}\langle -n\rangle).$$

Now we prove, as a corollary of Lemma 9.2.3:

Corollary 9.2.4. *The algebra B is non-negatively graded.*

Proof. We have to prove that $\mathrm{Hom}_{A,\mathbb{Z}}(P^{\mathrm{gr}}, P^{\mathrm{gr}}\langle n\rangle) = 0$ unless $n \leq 0$. So, let $n \in \mathbb{Z}$, and let $f : P^{\mathrm{gr}} \to P^{\mathrm{gr}}\langle n\rangle$ be a non-zero morphism. By Lemma 9.1.6(ii), the set

$$I := \{i \geq 0 \mid f(P^{\mathrm{gr}}) \subseteq \mathrm{rad}^{\mathrm{gr},i}(P^{\mathrm{gr}}\langle n\rangle)\}$$

is bounded above. Let $i = \max(I)$. Then f induces a non-zero morphism $g : P^{\mathrm{gr}} \to \big(\mathrm{rad}^{\mathrm{gr},i}(P^{\mathrm{gr}})/\mathrm{rad}^{\mathrm{gr},i+1}(P^{\mathrm{gr}})\big)\langle n\rangle$. By Lemma 9.2.3, $\mathrm{rad}^{\mathrm{gr},i}(P^{\mathrm{gr}})/\mathrm{rad}^{\mathrm{gr},i+1}(P^{\mathrm{gr}})$ is a direct sum of modules of the form $L_j^{\mathrm{gr}}\langle i\rangle$. As g is non-zero, we must have $n = -i$. This proves the result. □

The algebra B is finitely generated as a $\mathrm{S}(V)$-module, hence noetherian (even as a non-graded ring). If M is in $\mathrm{Mod}^{\mathrm{fg,gr}}(A)$, then $\mathrm{Hom}_A(P^{\mathrm{gr}}, M)$ is naturally a graded B-module (for all of this, see [AJS94, E.3]). By [AJS94, E.4] we have:

Proposition 9.2.5. *The functor*

$$\begin{cases} \mathrm{Mod}^{\mathrm{fg,gr}}(A) & \to & \mathrm{Mod}^{\mathrm{fg,gr}}(B) \\ M & \mapsto & \mathrm{Hom}_A(P^{\mathrm{gr}}, M) \end{cases}$$

is an equivalence of abelian categories.

Let us denote by S_i^{gr} the image of L_i^{gr} under this equivalence. The graded B-module S_i^{gr} is simple, concentrated in degree 0, and one-dimensional over $\Bbbk$. Applying the equivalence of Proposition 9.2.5 to property (9.2.2), one obtains:

$$\mathrm{Ext}^n_{B,\mathbb{Z}}(S_i^{\mathrm{gr}}, S_j^{\mathrm{gr}}\langle m\rangle) = 0 \quad \text{unless } n = m. \tag{9.2.6}$$

Lemma 9.2.7. *The (non-graded) ring B_0 is semi-simple.*

Proof. Let S_i be the image of S_i^{gr} under $\mathrm{For} : \mathrm{Mod}^{\mathrm{gr}}(B) \to \mathrm{Mod}(B)$. Using Corollary 9.2.4, the S_i are representatives of the simple B_0-modules. Hence it is sufficient to prove that for $i, j = 1 \ldots r$ we have $\mathrm{Ext}^1_{B_0}(S_i, S_j) = 0$. But if

$$0 \to S_j \to M \to S_i \to 0 \tag{9.2.8}$$

is a non-split B_0-extension, we can consider M as a graded B-module concentrated in degree 0, where B acts via the quotient $B/B_{>0} \cong B_0$. Then (9.2.8) yields a non-split graded B-extension of S_i^{gr} by S_j^{gr}, contradicting (9.2.6). □

Proposition 9.2.9. *B is a Koszul ring.*

Proof. This follows from [BGS96, 2.1.3], using Corollary 9.2.4, Lemma 9.2.7 and property (9.2.6). □

To conclude the proof of Theorem 9.2.1, we only have to compute $B^!$. The graded B-module B_0 is a direct sum of the simple modules S_i^{gr}, and for $i = 1 \dots n$, the module S_i^{gr} occurs with multiplicity $\dim_{\Bbbk}(\mathrm{Hom}_{B,\mathbb{Z}}(B_0, S_i^{\mathrm{gr}})) = \dim_{\Bbbk}(S_i^{\mathrm{gr}}) = 1$. Hence

$$(B^!)^{\mathrm{op}} \ = \ \bigoplus_n \mathrm{Ext}_B^n(B_0, B_0) \ \cong \ \bigoplus_{n,m} \mathrm{Ext}_{B,\mathbb{Z}}^n(\bigoplus_i S_i^{\mathrm{gr}}, \bigoplus_i S_i^{\mathrm{gr}}\langle m\rangle).$$

Using the equivalence of Proposition 9.2.5, we deduce

$$(B^!)^{\mathrm{op}} \ \cong \ \bigoplus_{n,m} \mathrm{Ext}_{A,\mathbb{Z}}^n(\bigoplus_i L_i^{\mathrm{gr}}, \bigoplus_i L_i^{\mathrm{gr}}\langle m\rangle) \ \cong \ \bigoplus_n \mathrm{Ext}_A^n(L, L).$$

9.3 First consequences of Theorem 4.4.3

We first consider the case $\lambda = 0$. We return to the setting of statement (‡) (see subsection 8.1), and choose the lifts $\mathcal{P}_w^{\mathrm{gr}}$, $P^{\mathrm{gr}}(w \bullet 0)$ and $\mathcal{L}_v^{\mathrm{gr}}$, $L^{\mathrm{gr}}(v \bullet 0)$ as in the statement. Let $v, w \in W^0$, and $i, j \in \mathbb{Z}$. We have a series of isomorphisms:

$$\begin{aligned}
& \mathrm{Hom}_{\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})^0)}\big(L^{\mathrm{gr}}(v \bullet 0), L^{\mathrm{gr}}(w \bullet 0)[i]\langle j\rangle\big) \\
\cong \ & \mathrm{Hom}_{\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})}\big(\zeta(\mathcal{L}_v^{\mathrm{gr}}), \zeta(\mathcal{L}_w^{\mathrm{gr}})[i+j]\langle j\rangle\big) \\
\cong \ & \mathrm{Hom}_{\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}^{(1)})}\big(\zeta(\mathcal{L}_v^{\mathrm{gr}}) \otimes \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho), \\
& \qquad\qquad \zeta(\mathcal{L}_w^{\mathrm{gr}}) \otimes \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho)[i+j]\langle j\rangle\big) \\
\cong \ & \mathrm{Hom}_{\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap} \mathcal{B})^{(1)})}\big(\kappa_{\mathcal{B}}(\zeta(\mathcal{L}_v^{\mathrm{gr}}) \otimes \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho)), \\
& \qquad\qquad \kappa_{\mathcal{B}}(\zeta(\mathcal{L}_w^{\mathrm{gr}}) \otimes \mathcal{O}_{\mathcal{B}^{(1)}}(-\rho))[i+j]\langle j\rangle\big) \\
\cong \ & \mathrm{Hom}_{\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap} \mathcal{B})^{(1)})}\big(\mathcal{P}_{\tau_0 v}^{\mathrm{gr}}, \mathcal{P}_{\tau_0 w}^{\mathrm{gr}}[i+j]\langle j\rangle\big) \\
\cong \ & \mathrm{Hom}_{\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})_0)}\big(P^{\mathrm{gr}}(\tau_0 v \bullet 0), P^{\mathrm{gr}}(\tau_0 w \bullet 0)[i+j]\langle j\rangle\big).
\end{aligned}$$

The first of these isomorphisms follows from Theorem 7.2.1 and Lemma 4.2.1. The second one is easy. The third isomorphism follows from the fact that $\kappa_{\mathcal{B}}$ is an equivalence (Theorem 2.3.11). The fourth one follows from (8.1.1). Finally, the fifth isomorphism follows from Theorem 6.3.4.

As the objects $P^{\mathrm{gr}}(-)$ are projective, from these isomorphisms we deduce:

Proposition 9.3.1. *Keep the assumptions of Theorem 4.4.3. Let $v, w \in W^0$, and $i, j \in \mathbb{Z}$. We have*

$$\mathrm{Hom}_{\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})^0)}(L^{\mathrm{gr}}(v \bullet 0), L^{\mathrm{gr}}(w \bullet 0)[i]\langle j\rangle) = 0 \quad \textit{unless } i = -j.$$

Using the isomorphisms

$$\bigoplus_{i\geq 0} \mathrm{Ext}^i_{(\mathcal{U}\mathfrak{g})^0}(L(v\bullet 0), L(w\bullet 0)) \;\cong\; \bigoplus_{i,j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})^0)}(L^{\mathrm{gr}}(v\bullet 0), L^{\mathrm{gr}}(w\bullet 0)[i]\langle j\rangle)$$

and

$$\mathrm{Hom}_{(\mathcal{U}\mathfrak{g})_0}(P(v\bullet 0), P(w\bullet 0)) \;\cong\; \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})_0}(P^{\mathrm{gr}}(v\bullet 0), P^{\mathrm{gr}}(w\bullet 0)\langle j\rangle),$$

we also deduce the following:

Proposition 9.3.2. *Keep the assumptions of Theorem* 4.4.3.

(i) *Let $v, w\in W^0$. There exists an isomorphism*

$$\bigoplus_{i\geq 0} \mathrm{Ext}^i_{(\mathcal{U}\mathfrak{g})^0}(L(v\bullet 0), L(w\bullet 0)) \;\cong\; \mathrm{Hom}_{(\mathcal{U}\mathfrak{g})_0}(P(\tau_0 v\bullet 0), P(\tau_0 w\bullet 0)).$$

(ii) *Let $L := \bigoplus_{w\in W^0} L(w\bullet 0)$ and $P := \bigoplus_{w\in W^0} P(w\bullet 0)$. There exists an isomorphism of algebras*

$$\bigoplus_{i\geq 0} \mathrm{Ext}^i_{(\mathcal{U}\mathfrak{g})^0}(L, L) \;\cong\; \mathrm{End}_{(\mathcal{U}\mathfrak{g})_0}(P).$$

9.4 The ring $A_{\widetilde{\mathcal{N}}}$

Recall the vector bundle $\mathcal{M}^0$ on the formal neighborhood of $\mathcal{B}^{(1)}$ in $\widetilde{\mathfrak{g}}^{(1)}$ defined in subsection I.1.2 (here we use the identification of this formal neighborhood with the formal neighborhood of $\mathcal{B}^{(1)}\times\{0\}$ in $\widetilde{\mathfrak{g}}^{(1)}\times_{\mathfrak{h}^{*(1)}}\mathfrak{h}^*$). Let $\mathcal{M}^0_0$ be the restriction of $\mathcal{M}^0$ to the formal neighborhood of $\mathcal{B}^{(1)}$ in $\widetilde{\mathcal{N}}^{(1)}$. This is the splitting bundle involved in the definition of equivalence $\epsilon_0^{\mathcal{B}}$.

In [BM] (see also [Bez06b]), the authors prove the following:

Theorem 9.4.1. *There exists a vector bundle $\mathcal{M}_{\widetilde{\mathcal{N}}}$ on $\widetilde{\mathcal{N}}^{(1)}$, whose restriction to the formal neighborhood of $\mathcal{B}^{(1)}$ is isomorphic to $\mathcal{M}^0_0$. Moreover, this vector bundle can be endowed with a $\mathbb{G}_{\mathbf{m}}$-equivariant structure, compatible with the action defined in* (5.2.1).

Let us consider the algebra

$$A_{\widetilde{\mathcal{N}}} := \Gamma(\widetilde{\mathcal{N}}^{(1)}, \mathcal{E}nd_{\mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}}(\mathcal{M}_{\widetilde{\mathcal{N}}})).$$

This a $\mathrm{S}(\mathfrak{g}^{(1)})$-algebra, finitely generated as a $\mathrm{S}(\mathfrak{g}^{(1)})$-module (because the natural morphism $\widetilde{\mathcal{N}}^{(1)}\to\mathfrak{g}^{*(1)}$ is proper). For any $\mathrm{S}(\mathfrak{g}^{(1)})$-algebra A, we denote by $\mathrm{Mod}_0^{\mathrm{fg}}(A)$ the category of finitely generated A-modules, on which the image of $\mathfrak{g}^{(1)}$ acts nilpotently. By definition we have an equivalence of categories

$$\mathrm{Mod}_0^{\mathrm{fg}}(A_{\widetilde{\mathcal{N}}}) \;\cong\; \mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^0). \tag{9.4.2}$$

9.5 Koszulity of regular blocks of $(\mathcal{U}\mathfrak{g})_0$

One of the main results of this chapter is the following:

Theorem 9.5.1. *Assume $p > h$ is large enough so that Lusztig's conjecture is true, and let $\lambda \in \mathbb{X}$ be regular.*

There exists a Koszul ring $B_{\mathcal{B}}$, which is naturally a $\mathrm{S}(\mathfrak{g}^{(1)})$-algebra, and equivalences of categories

$$\begin{aligned} \mathrm{Mod}_0^{\mathrm{fg}}(B_{\mathcal{B}}) &\cong \mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^{\lambda}), \\ \mathrm{Mod}^{\mathrm{fg}}((B_{\mathcal{B}})^!) &\cong \mathrm{Mod}_{\lambda}^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0). \end{aligned}$$

In particular, the ring $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$ can be endowed with a Koszul grading.

Remark 9.5.2. The fact that the category $\mathrm{Mod}_{\lambda}^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$ is equivalent to the category of (non-graded) modules over a Koszul ring was proved in [AJS94, 18.21]. Their proof relies on an explicit computation of the Poincaré polynomial of $(\mathcal{U}\mathfrak{g})_0^{\hat{\lambda}}$. The fact that the dual Koszul ring "controls" the category $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^{\lambda})$ is new, however.

Proof of Theorem 9.5.1. Let us consider the first statement. As C_0 is a fundamental domain for the action of W_{aff} on the set of regular integral weights, we can assume $\lambda \in C_0$. Then, as the category $\mathrm{Mod}_{\lambda}^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$ (and, similarly, $\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^{\lambda})$) does not depend, up to equivalence, on the choice of $\lambda \in C_0$ (use translation functors), we can assume $\lambda = 0$.

By Theorem 9.4.1, the algebra $A_{\widetilde{\mathcal{N}}}$ can be endowed with a grading. Let $A_{\widetilde{\mathcal{N}}}^+$ be $A_{\widetilde{\mathcal{N}}}$ with the grading provided by this theorem. We define the category $\mathrm{Mod}_0^{\mathrm{fg,gr}}(A_{\widetilde{\mathcal{N}}}^+)$ as above. The choice of the $\mathbb{G}_{\mathbf{m}}$-equivariant structure in subsection 7.2 was arbitrary. From now on we choose as this structure the restriction of the $\mathbb{G}_{\mathbf{m}}$-equivariant structure of Theorem 9.4.1. Then we have by definition an equivalence

$$\mathrm{Mod}_0^{\mathrm{fg,gr}}(A_{\widetilde{\mathcal{N}}}^+) \cong \mathrm{Mod}_0^{\mathrm{fg,gr}}((\mathcal{U}\mathfrak{g})^0). \tag{9.5.3}$$

Now, let $A_{\widetilde{\mathcal{N}}}^-$ be $A_{\widetilde{\mathcal{N}}}$ with the opposite grading, defined by $(A_{\widetilde{\mathcal{N}}}^-)_n := (A_{\widetilde{\mathcal{N}}}^+)_{-n}$. This algebra is a finite $\mathrm{S}(\mathfrak{g}^{(1)})$-algebra, where $\mathfrak{g}^{(1)}$ is in degree 2. There is a natural equivalence of categories

$$\mathrm{Mod}^{\mathrm{gr}}(A_{\widetilde{\mathcal{N}}}^+) \cong \mathrm{Mod}^{\mathrm{gr}}(A_{\widetilde{\mathcal{N}}}^-) \tag{9.5.4}$$

sending a graded module to the module with the opposite grading. Hence, using equivalence (9.5.3) together with Proposition 9.3.1, the assumptions of Theorem 9.2.1 are satisfied by the graded ring $A_{\widetilde{\mathcal{N}}}^-$. It follows that there exists a Koszul ring $B_{\mathcal{B}}$, Morita equivalent to $A_{\widetilde{\mathcal{N}}}^-$. By construction, using equivalence (9.4.2), the first equivalence of the theorem is satisfied.

Again by Theorem 9.2.1 and equivalence (9.4.2), with the notation of Proposition 9.3.2, the dual ring $(B_{\mathcal{B}})^!$ is isomorphic to

$$\Big(\bigoplus_n \mathrm{Ext}^n_{(\mathcal{U}\mathfrak{g})^0}(L, L)\Big)^{\mathrm{op}}.$$

(Here we have also used [BMR08, 3.1.7] to identify the Ext groups in the different categories.) By Proposition 9.3.2(ii), this ring is isomorphic (as a non-graded ring) to the ring $(\mathrm{End}_{(\mathcal{U}\mathfrak{g})_0}(P))^{\mathrm{op}}$, which is Morita equivalent to $(\mathcal{U}\mathfrak{g})_0^{\hat{0}}$. This gives the second equivalence.

Finally, the second assertion of the theorem follows from the second equivalence (and the fact that $B^!_{\mathcal{B}}$ is Koszul), using [AJS94, F.3]. $\square$

10 Parabolic analogues: Koszulity of singular blocks of $(\mathcal{U}\mathfrak{g})_0$

In this section we extend the main results of sections 8 and 9 to the case of a singular weight.

10.1 Review of some results of [BMR06]

Let $P \subset G$ be a standard parabolic subgroup, and $\mathcal{P} := G/P$ be the associated flag variety. Let $\mathfrak{p}$ be the Lie algebra of P, let ρ_P be the half sum of the positive roots of the Levi of P, and let $N_{\mathcal{P}} := \dim(\mathcal{P})$. Recall the variety $\widetilde{\mathfrak{g}}_{\mathcal{P}}$ introduced in subsection I.1.2. Let us also consider the variety

$$\widetilde{\mathcal{N}}_{\mathcal{P}} := T^*\mathcal{P} = \{(X, gP) \in \mathfrak{g}^* \times \mathcal{P} \mid X_{|g \cdot \mathfrak{p}} = 0\}.$$

We have already considered this variety in (7.3.1) in the special case $P = P_\alpha$. Under the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, $\widetilde{\mathfrak{g}}_{\mathcal{P}}$ identifies with the orthogonal of $\widetilde{\mathcal{N}}_{\mathcal{P}}$ in $\mathfrak{g}^* \times \mathcal{P}$. Hence we have a Koszul duality (see Theorem 2.3.11):

$$\kappa_{\mathcal{P}} : \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)}) \ \xrightarrow{\sim} \ \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}).$$

In this subsection we give a representation-theoretic interpretation of both of these categories. First, choose a weight $\mu \in \mathbb{X}$, on the reflection hyperplanes corresponding to the parabolic P, and not on any other reflection hyperlane (for W_{aff}). A particular case of Theorem 3.3.15 gives an equivalence of categories

$$\widehat{\gamma}_{\mu}^{\mathcal{P}} : \mathrm{DGCoh}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) \ \xrightarrow{\sim} \ \mathcal{D}^b\mathrm{Mod}^{\mathrm{fg}}_{\mu}((\mathcal{U}\mathfrak{g})_0).$$

The representation-theoretic interpretation of $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)})$ is given by the results of [BMR06, 1.10]. Let $\mathbb{X}_{\mathcal{P}}$ be the sublattice of $\mathbb{X}$ consisting of the $\lambda \in \mathbb{X}$ such that $\langle \lambda, \alpha^\vee \rangle = 0$ for any root α of the Levi of P. For $\lambda \in \mathbb{X}_{\mathcal{P}}$, let $\mathfrak{D}^{\lambda}_{\mathcal{P}} := \mathcal{O}_{\mathcal{P}}(\lambda) \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{D}_{\mathcal{P}} \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}_{\mathcal{P}}(-\lambda)$ be the sheaf of twisted differential operators on $\mathcal{P}$ (as in *loc. cit.*). Let λ be a *regular weight* in $\mathbb{X}_{\mathcal{P}}$. We will assume[13] that

$$R^i\Gamma(\mathfrak{D}^{\lambda}_{\mathcal{P}}) = 0 \quad \text{for } i > 0. \tag{10.1.1}$$

Then we define

$$U^{\lambda}_{\mathcal{P}} := \Gamma(\mathfrak{D}^{\lambda}_{\mathcal{P}}).$$

[13]This condition is satisfied in particular if $\mathrm{char}(\Bbbk)$ is greater than an explicitly computable bound depending on G and λ (see [BMR06, 1.10.9(ii)]).

We denote by $\mathrm{Mod}_0^{\mathrm{fg}}(U_{\mathcal{P}}^{\lambda})$ the category of finitely generated $U_{\mathcal{P}}^{\lambda}$-modules on which the central subalgebra $\Gamma(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)}, \mathcal{O}_{\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)}})$ (the image of the center of $\mathfrak{D}_{\mathcal{P}}^{\lambda}$) acts with trivial generalized character. By [BMR06, 1.10.4] we have:

Theorem 10.1.2. *Assume* (10.1.1) *is satisfied. There exists an equivalence of categories*

$$\mathcal{D}^b\mathrm{Coh}_{\mathcal{P}^{(1)}}(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)}) \xrightarrow{\sim} \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}(U_{\mathcal{P}}^{\lambda}).$$

This theorem gives a representation-theoretic interpretation for $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)})$. As in Theorem I.1.2.1, the equivalence of Theorem 10.1.2 depends on the choice of a splitting bundle. We choose it as in [BMR06, 1.10.3], and denote by $\Upsilon_{\lambda}^{\mathcal{P}}$ the equivalence associated to λ. Let us remark that for $\mathcal{P} = \mathcal{B}$ we have $U_{\mathcal{B}}^{\lambda} = (\mathcal{U}\mathfrak{g})^{\lambda}$, but $\Upsilon_{\lambda}^{\mathcal{B}} = \epsilon_{\lambda - p\rho}^{\mathcal{B}}$ (see [BMR06, 1.10.5], and compare *e.g.* with the proof of Lemma I.1.4.1). We deduce (see the formula at the end of I.1.2):

$$\Upsilon_{\lambda}^{\mathcal{B}}(\mathcal{F}) = \epsilon_{\lambda}^{\mathcal{B}}(\mathcal{F} \otimes_{\mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}} \mathcal{O}_{\widetilde{\mathfrak{g}}^{(1)}}(-\rho)). \tag{10.1.3}$$

There is a natural morphism of algebras $\phi_{\mathcal{P}}^{\lambda} : (\mathcal{U}\mathfrak{g})^{\lambda} \to U_{\mathcal{P}}^{\lambda}$, coming from the action of G on $\mathcal{P}$ (see [BMR06, 1.10.7]). We denote by $(\phi_{\mathcal{P}}^{\lambda})^* : \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}(U_{\mathcal{P}}^{\lambda}) \to \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^{\lambda})$ the corresponding "restriction" functor. Consider the diagram

$$\widetilde{\mathcal{N}} \xleftarrow{j_{\mathcal{P}}} \widetilde{\mathcal{N}}_{\mathcal{P}} \times_{\mathcal{P}} \mathcal{B} \xrightarrow{\rho_{\mathcal{P}}} \widetilde{\mathcal{N}}_{\mathcal{P}},$$

where $j_{\mathcal{P}}$ is the natural embedding, and $\rho_{\mathcal{P}}$ is induced by the projection $\pi_{\mathcal{P}} : \mathcal{B} \to \mathcal{P}$. Then by [BMR06, 1.10.7] the following holds:

Proposition 10.1.4. *The following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{D}^b\mathrm{Coh}_{\mathcal{P}^{(1)}}(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)}) \arrow[r, "\Upsilon_{\lambda}^{\mathcal{P}}", "\sim"'] \arrow[d, "(j_{\mathcal{P}})_*(\rho_{\mathcal{P}})^*"'] & \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}(U_{\mathcal{P}}^{\lambda}) \arrow[d, "(\phi_{\mathcal{P}}^{\lambda})^*"] \\
\mathcal{D}^b\mathrm{Coh}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}) \arrow[r, "\Upsilon_{\lambda}^{\mathcal{B}}", "\sim"'] & \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^{\lambda}).
\end{tikzcd}$$

10.2 Koszul duality for singular blocks

We choose λ and μ as in subsection 10.1, and assume moreover that μ is in the closure of the alcove of λ. Let $y \in W_{\mathrm{aff}}$ be the unique element such that $\lambda_0 := y^{-1} \bullet \lambda \in C_0$. Then $\mu_0 := y^{-1} \bullet \mu \in \overline{C_0}$.

For simplicity, in what follows we make the following assumption[14]:

$$\phi_{\mathcal{P}}^{\lambda} \text{ is surjective.} \tag{10.2.1}$$

[14]In [BMR06, 1.10.9] it is proved that this assumption is satisfied when $\mathrm{char}(\Bbbk)$ is greater than an explicit bound depending on G and λ and, moreover, a sufficient condition is given for this to be satisfied in arbitrary characteristic. The latter condition is satisfied if $G = \mathrm{SL}(n, \Bbbk)$ (see [Hum95, 5.5] and [Don90] or [MvdK92]) or if $P = P_{\{\alpha\}}$ for a short simple root α (see [BK04, 5.3]).

It follows from this fact that if L is a simple $U^{\lambda}_{\mathcal{P}}$-module then $(\phi^{\lambda}_{\mathcal{P}})^* L$ is a simple $(\mathcal{U}\mathfrak{g})^{\lambda}$-module. Hence, if L has trivial central character, then $(\phi^{\lambda}_{\mathcal{P}})^* L \cong L(w \bullet \lambda_0)$ for a unique $w \in W^0$ (see subsection 4.4). In this case, by definition we set $L = L_{\mathcal{P}}(w \bullet \lambda_0)$. We denote by I_λ the set of $w \in W^0$ such that $L_{\mathcal{P}}(w \bullet \lambda_0)$ is defined.

Let $W^0_\mu \subset W^0$ by the subset of elements $w \in W^0$ such that $w \bullet \mu_0$ is in the upper closure of $w \bullet C_0$. As in subsection 4.4, $\mathrm{Mod}^{\mathrm{fg}}_{\mu}((\mathcal{U}\mathfrak{g})_0)$ is the category of finitely generated modules over the algebra $(\mathcal{U}\mathfrak{g})^{\hat{\mu}}_0$ (the block of $(\mathcal{U}\mathfrak{g})_0$ associated to μ). The simple objects in this category are the image of the simple G-modules $L(w \bullet \mu_0)$ for $w \in W^0_\mu$. We denote by $P(w \bullet \mu_0)$ the projective cover of $L(w \bullet \mu_0)$.

It is not clear *a priori* how to determine I_λ in general; this will be part of Theorem 10.2.4 below. However, let us remark already that

$$\# I_\lambda \;=\; \# W^0_\mu. \tag{10.2.2}$$

Indeed, the left hand side is the rank of the Grothendieck group $K^0(\mathrm{Mod}^{\mathrm{fg}}_0(U^{\lambda}_{\mathcal{P}}))$, which is isomorphic, by Theorem 10.1.2, to $K^0(\mathrm{Coh}_{\mathcal{P}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}_{\mathcal{P}})) \cong K(\mathcal{P})$, while the right hand side is the rank of $K^0(\mathrm{Mod}^{\mathrm{fg}}_{\mu}((\mathcal{U}\mathfrak{g})_0))$, which is isomorphic to $K^0(\mathrm{Mod}^{\mathrm{fg}}_{(0,\mu)}(\mathcal{U}\mathfrak{g}))$, hence, by Theorem I.1.2.1, to $K^0(\mathrm{Coh}_{\mathcal{P}^{(1)}}(\widetilde{\mathfrak{g}}^{(1)}_{\mathcal{P}})) \cong K(\mathcal{P})$.

As in subsection 6.3, the algebra $(\mathcal{U}\mathfrak{g})^{\hat{\mu}}_0$ can be endowed with a grading, and there exists a fully faithful triangulated functor commuting with internal shifts

$$\widetilde{\gamma}^{\mathcal{P}}_{\mu} : \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) \;\to\; \mathcal{D}^b \mathrm{Mod}^{\mathrm{fg,gr}}_{\mu}((\mathcal{U}\mathfrak{g})_0),$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) \arrow[r, "\widetilde{\gamma}^{\mathcal{P}}_{\mu}"] \arrow[d, "\mathrm{For}"'] & \mathcal{D}^b \mathrm{Mod}^{\mathrm{fg,gr}}_{\mu}((\mathcal{U}\mathfrak{g})_0) \arrow[d, "\mathrm{For}"] \\
\mathrm{DGCoh}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) \arrow[r, "\widetilde{\gamma}^{\mathcal{P}}_{\mu}", "\sim"'] & \mathcal{D}^b \mathrm{Mod}^{\mathrm{fg}}_{\mu}((\mathcal{U}\mathfrak{g})_0).
\end{tikzcd}$$

One can lift the projective modules $P(w \bullet \mu_0)$ to graded $(\mathcal{U}\mathfrak{g})^{\hat{\mu}}_0$-modules (uniquely, up to a shift; see Theorem 5.6.1). Moreover, we have:

Lemma 10.2.3. *The functor $\widetilde{\gamma}^{\mathcal{P}}_{\mu}$ is an equivalence of categories. In particular, the lifts of the projective modules $P(w \bullet \mu_0)$ ($w \in W^0_\mu$) are in the essential image of $\widetilde{\gamma}^{\mathcal{P}}_{\mu}$.*

Proof. It is enough to prove that the lifts of the simple $(\mathcal{U}\mathfrak{g})^{\hat{\mu}}_0$-modules are in the essential image of $\widetilde{\gamma}^{\mathcal{P}}_{\mu}$.

Let $\nu \in y \bullet C_0$, and let $\nu_0 = y^{-1} \bullet \nu$. The simple $(\mathcal{U}\mathfrak{g})^{\hat{\mu}}_0$-modules are in the essential image of the translation functor $T^{\mu}_{\nu} : \mathrm{Mod}^{\mathrm{fg}}_{\nu}((\mathcal{U}\mathfrak{g})_0) \to \mathrm{Mod}^{\mathrm{fg}}_{\mu}((\mathcal{U}\mathfrak{g})_0)$. More precisely, for $w \in W^0_\mu$ we have $L(w \bullet \mu_0) = T^{\mu}_{\nu} L(w \bullet \nu_0)$. Moreover, by Proposition 5.4.2, we have an isomorphism of functors $\widehat{\gamma}_{\mu}^{\mathcal{P}} \circ R(\widehat{\pi}_{\mathcal{P}})_* \cong T_{\nu}^{\mu} \circ \widehat{\gamma}_{\nu}^{\mathcal{B}}$. The functor $R(\widehat{\pi}_{\mathcal{P}})_*$ has a natural graded version, the functor

$$R(\widehat{\pi}_{\mathcal{P},\mathbb{G}_{\mathbf{m}}})_* : \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}) \ \to \ \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}).$$

The functor $\widehat{\gamma}_{\nu}^{\mathcal{B}}$ has a "graded version" $\widetilde{\gamma}_{\nu}^{\mathcal{B}}$ (see subsection 8.5) which, by Remark 6.3.5, is an equivalence of categories. If, for $w \in W_{\mu}^{0}$, $\mathcal{M}_w$ is the inverse image under $\widetilde{\gamma}_{\nu}^{\mathcal{B}}$ of a lift of $L(w \bullet \nu_0)$, then one easily checks that $R(\widehat{\pi}_{\mathcal{P},\mathbb{G}_{\mathbf{m}}})_* \mathcal{M}_w$ is sent by $\widetilde{\gamma}_{\mu}^{\mathcal{P}}$ to a lift of the simple module $L(w \bullet \mu_0) \in \mathrm{Mod}_{\mu}^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0)$. This concludes the proof. $\square$

Similarly, as in subsection 7.2, the completion of the algebra $U_{\mathcal{P}}^{\lambda}$ with respect to the trivial central character can be endowed with a $\mathbb{G}_{\mathbf{m}}$-equivariant structure, and there exists a fully faithful functor commuting with internal shifts

$$\widetilde{\Upsilon}_{\lambda}^{\mathcal{P}} : \mathcal{D}^b \mathrm{Coh}_{\mathcal{P}^{(1)}}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)}) \ \to \ \mathcal{D}^b \mathrm{Mod}_0^{\mathrm{fg,gr}}(U_{\mathcal{P}}^{\lambda}),$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}^b \mathrm{Coh}_{\mathcal{P}^{(1)}}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)}) \arrow[r, "\widetilde{\Upsilon}_{\lambda}^{\mathcal{P}}"] \arrow[d, "\mathrm{For}"] & \mathcal{D}^b \mathrm{Mod}_0^{\mathrm{fg,gr}}(U_{\mathcal{P}}^{\lambda}) \arrow[d, "\mathrm{For}"] \\
\mathcal{D}^b \mathrm{Coh}_{\mathcal{P}^{(1)}}(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)}) \arrow[r, "\Upsilon_{\lambda}^{\mathcal{P}}", "\sim"'] & \mathcal{D}^b \mathrm{Mod}_0^{\mathrm{fg}}(U_{\mathcal{P}}^{\lambda}).
\end{tikzcd}$$

The simple objects in the category $\mathrm{Mod}_0^{\mathrm{fg}}(U_{\mathcal{P}}^{\lambda})$ are the $L_{\mathcal{P}}(w \bullet \lambda_0)$ for $w \in I_{\lambda}$. They can be lifted to graded modules (uniquely, up to a shift). We will prove below that the lifts of the simple modules are in the essential image of $\widetilde{\Upsilon}_{\lambda}^{\mathcal{P}}$. In particular, this functor is an equivalence.

Finally, as in subsection 4.2, there exists a fully faithful functor

$$\zeta_{\mathcal{P}} : \mathcal{D}^b \mathrm{Coh}_{\mathcal{P}^{(1)}}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)}) \ \to \ \mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)})$$

with the same properties as ζ.

The following theorem is a "parabolic analogue" of Theorem 4.4.3.

Theorem 10.2.4. *Assume $p > h$ is large enough so that Lusztig's conjecture is true. Assume moreover that* (10.1.1) *and* (10.2.1) *are satisfied.*

(i) *We have $I_{\lambda} = \tau_0 W_{\mu}^{0}$, and the lifts of the simple modules are in the essential image of $\widetilde{\Upsilon}_{\lambda}^{\mathcal{P}}$.*

(ii) *There is a unique choice of the lifts*[15] $P^{\mathrm{gr}}(v \bullet \mu_0)$ *($v \in W_{\mu}^{0}$),* $L_{\mathcal{P}}^{\mathrm{gr}}(u \bullet \lambda_0)$ *($u \in I_{\lambda}$) such that, if $\mathcal{Q}_{\mathcal{P},v}^{y,\mathrm{gr}}$, resp. $\mathcal{L}_{\mathcal{P},u}^{y,\mathrm{gr}}$ is the object of* $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)})$*, respectively* $\mathcal{D}^b \mathrm{Coh}_{\mathcal{P}^{(1)}}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)})$*, such that $P^{\mathrm{gr}}(v \bullet \mu_0) \cong \widetilde{\gamma}_{\mu}^{\mathcal{P}}(\mathcal{Q}_{\mathcal{P},v}^{y,\mathrm{gr}})$, respectively $L_{\mathcal{P}}^{\mathrm{gr}}(u \bullet \lambda_0) \cong \widetilde{\Upsilon}_{\lambda}^{\mathcal{P}}(\mathcal{L}_{\mathcal{P},u}^{y,\mathrm{gr}})$, for all $w \in W_{\mu}^{0}$ we have in* $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)})$*:*

$$\kappa_{\mathcal{P}}^{-1} \mathcal{Q}_{\mathcal{P},w}^{y,\mathrm{gr}} \ \cong \ \zeta_{\mathcal{P}}(\mathcal{L}_{\mathcal{P},\tau_0 w}^{y,\mathrm{gr}}) \otimes_{\mathcal{O}_{\mathcal{P}^{(1)}}} \mathcal{O}_{\mathcal{P}^{(1)}}(2\rho_P - 2\rho). \tag{10.2.5}$$

[15] *A priori*, these lifts depend on the choice of λ, μ, *i.e.* on y, although it does not appear in the notation.

Proof. We prove (i) and (ii) simultaneously. We choose the objects $\mathcal{P}_w^{y,\mathrm{gr}}$, $\mathcal{L}_w^{y,\mathrm{gr}}$ ($w \in W^0$) as in subsection 8.5 (hence as in Theorem 4.4.3 if $y = 1$). Here, to avoid confusion, we change the notation $\mathcal{P}_w^{y,\mathrm{gr}}$ in $\mathcal{Q}_w^{y,\mathrm{gr}}$. As for Theorem 4.4.3, the unicity statement is easy to prove, and we concentrate on the existence of the lifts.

As above (and in subsection I.1.1), let $\widetilde{\pi}_{\mathcal{P}} : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}_{\mathcal{P}}$ be the natural morphism. It induces a morphism of dg-schemes (see (5.4.1))

$$\widehat{\pi}_{\mathcal{P}} : (\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)} \to (\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}.$$

By Proposition 5.4.2, we have an isomorphism of functors

$$T_\mu^\lambda \circ \widehat{\gamma}_\mu^{\mathcal{P}} \;\cong\; \widehat{\gamma}_\lambda^{\mathcal{B}} \circ L(\widehat{\pi}_{\mathcal{P}})^*. \tag{10.2.6}$$

By adjunction, and using equation (4.3.2), we have for $w \in W_\mu^0$:

$$T_\mu^\lambda P(w \bullet \mu_0) \;\cong\; P(w \bullet \lambda_0). \tag{10.2.7}$$

The functor $L(\widehat{\pi}_{\mathcal{P}})^*$ has a natural graded version, the functor

$$L(\widehat{\pi}_{\mathcal{P},\mathbb{G}_\mathbf{m}})^* : \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}) \;\to\; \mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)}).$$

For $w \in W_\mu^0$, we define $P^{\mathrm{gr}}(w \bullet \mu_0)$ as the unique lift of $P(w \bullet \mu_0)$ such that, if $\mathcal{Q}_{\mathcal{P},w}^{y,\mathrm{gr}}$ is the object of $\mathrm{DGCoh}^{\mathrm{gr}}((\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)})$ such that $P^{\mathrm{gr}}(w \bullet \mu_0) \cong \widetilde{\gamma}_\mu^{\mathcal{P}}(\mathcal{Q}_{\mathcal{P},w}^{y,\mathrm{gr}})$ (such an object exists by Lemma 10.2.3), we have

$$\mathcal{Q}_w^{y,\mathrm{gr}}\langle N - N_{\mathcal{P}} \rangle \;\cong\; L(\widehat{\pi}_{\mathcal{P},\mathbb{G}_\mathbf{m}})^* \mathcal{Q}_{\mathcal{P},w}^{y,\mathrm{gr}}. \tag{10.2.8}$$

Such a lift exists thanks to isomorphisms (10.2.6) and (10.2.7).

The morphisms $j_{\mathcal{P}}$ and $\rho_{\mathcal{P}}$ induce functors

$$\begin{aligned}(j_{\mathcal{P},\mathbb{G}_\mathbf{m}})_* : \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}_{\mathcal{B}^{(1)}}((\widetilde{\mathcal{N}}_{\mathcal{P}} \times_{\mathcal{P}} \mathcal{B})^{(1)}) \;&\to\; \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}_{\mathcal{B}^{(1)}}(\widetilde{\mathcal{N}}^{(1)}),\\ (\rho_{\mathcal{P},\mathbb{G}_\mathbf{m}})^* : \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}_{\mathcal{P}^{(1)}}(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)}) \;&\to\; \mathcal{D}^b\mathrm{Coh}^{\mathbb{G}_\mathbf{m}}_{\mathcal{B}^{(1)}}((\widetilde{\mathcal{N}}_{\mathcal{P}} \times_{\mathcal{P}} \mathcal{B})^{(1)}).\end{aligned}$$

Consider the following factorization of $\widetilde{\pi}_{\mathcal{P}}$:

$$\widetilde{\mathfrak{g}} \xhookrightarrow{\widetilde{\pi}_{\mathcal{P},1}} \widetilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathcal{P}} \mathcal{B} \xrightarrow{\widetilde{\pi}_{\mathcal{P},2}} \widetilde{\mathfrak{g}}_{\mathcal{P}},$$

where $\widetilde{\pi}_{\mathcal{P},2}$ is induced by the projection $\pi_{\mathcal{P}}$. These morphisms induce

$$\begin{aligned}\widehat{\pi}_{\mathcal{P},1} : (\widetilde{\mathfrak{g}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)} \;&\to\; ((\widetilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathcal{P}} \mathcal{B}) \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)},\\ \widehat{\pi}_{\mathcal{P},2} : ((\widetilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathcal{P}} \mathcal{B}) \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{B}} \mathcal{B})^{(1)} \;&\to\; (\widetilde{\mathfrak{g}}_{\mathcal{P}} \overset{R}{\cap}_{\mathfrak{g}^* \times \mathcal{P}} \mathcal{P})^{(1)}.\end{aligned}$$

Then we have

$$L(\widehat{\pi}_{\mathcal{P},\mathbb{G}_\mathbf{m}})^* \;\cong\; L(\widehat{\pi}_{\mathcal{P},1,\mathbb{G}_\mathbf{m}})^* \circ L(\widehat{\pi}_{\mathcal{P},2,\mathbb{G}_\mathbf{m}})^*.$$

Using this equality and the results of subsections 2.4 and 2.5, one can identify the Koszul dual (with respect to $\kappa_{\mathcal{B}}$, $\kappa_{\mathcal{P}}$) of the functor $L(\widehat{\pi}_{\mathcal{P},\mathbb{G}_{\mathbf{m}}})^*$. Namely, a proof similar to that of Theorem 8.2.1 gives an isomorphism

$$\begin{aligned}(\kappa_{\mathcal{B}})^{-1} \circ L(\widehat{\pi}_{\mathcal{P},\mathbb{G}_{\mathbf{m}}})^* \circ \kappa_{\mathcal{P}} \ \cong \\ \big(R(\widetilde{j_{\mathcal{P}\mathbb{G}_{\mathbf{m}}}})_* \circ L(\widetilde{\rho_{\mathcal{P}\mathbb{G}_{\mathbf{m}}}})^*\big) \otimes_{\mathcal{B}^{(1)}} \mathcal{O}_{\mathcal{B}^{(1)}}(-2\rho_P)[N-N_{\mathcal{P}}]\langle 2(N-N_{\mathcal{P}})\rangle,\end{aligned} \quad (10.2.9)$$

where the functors $R(\widetilde{j_{\mathcal{P}\mathbb{G}_{\mathbf{m}}}})_*$ and $L(\widetilde{\rho_{\mathcal{P}\mathbb{G}_{\mathbf{m}}}})^*$ are defined as in 2.4 and 2.5.

Now we have introduced all the tools needed for the proof of Theorem 10.2.4. Let $w \in W_\mu^0$. Consider the object

$$\mathcal{F}_w := \big(\kappa_{\mathcal{P}}^{-1} \mathcal{Q}_{\mathcal{P},w}^{y,\mathrm{gr}}\big) \otimes_{\mathcal{O}_{\mathcal{P}^{(1)}}} \mathcal{O}_{\mathcal{P}^{(1)}}(2\rho - 2\rho_P)$$

of $\mathrm{DGCoh}^{\mathrm{gr}}(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)})$. By equation (10.2.9) we have

$$\begin{aligned}\big(R(\widetilde{j_{\mathcal{P}\mathbb{G}_{\mathbf{m}}}})_* \circ L(\widetilde{\rho_{\mathcal{P}\mathbb{G}_{\mathbf{m}}}})^*\big)(\mathcal{F}_w) \ \cong \\ (\kappa_{\mathcal{B}})^{-1} \circ L(\widehat{\pi}_{\mathcal{P},\mathbb{G}_{\mathbf{m}}})^*(\mathcal{Q}_{\mathcal{P},w}^{y,\mathrm{gr}} \otimes_{\mathcal{P}^{(1)}} \mathcal{O}_{\mathcal{P}^{(1)}}(2\rho - 2\rho_P)) \\ \otimes_{\mathcal{B}^{(1)}} \mathcal{O}_{\mathcal{B}^{(1)}}(2\rho_P)[N_{\mathcal{P}} - N]\langle 2(N_{\mathcal{P}} - N)\rangle.\end{aligned}$$

Using definition (10.2.8) we deduce

$$\big(R(\widetilde{j_{\mathcal{P}\mathbb{G}_{\mathbf{m}}}})_* \circ L(\widetilde{\rho_{\mathcal{P}\mathbb{G}_{\mathbf{m}}}})^*\big)(\mathcal{F}_w) \ \cong \ (\kappa_{\mathcal{B}})^{-1}(\mathcal{Q}_w^{y,\mathrm{gr}}) \otimes_{\mathcal{B}^{(1)}} \mathcal{O}_{\mathcal{B}^{(1)}}(2\rho)[N_{\mathcal{P}} - N]\langle N_{\mathcal{P}} - N\rangle.$$

Finally, by (8.1.1) (or its analogue in subsection 8.5 if $y \neq 1$) we have

$$\big(R(\widetilde{j_{\mathcal{P}\mathbb{G}_{\mathbf{m}}}})_* \circ L(\widetilde{\rho_{\mathcal{P}\mathbb{G}_{\mathbf{m}}}})^*\big)(\mathcal{F}_w) \ \cong \ \zeta(\mathcal{L}_{\tau_0 w}^{y,\mathrm{gr}}\langle N_{\mathcal{P}} - N\rangle) \otimes_{\mathcal{B}^{(1)}} \mathcal{O}_{\mathcal{B}^{(1)}}(\rho).$$

We deduce easily that there exists an object $\mathcal{G}_w$ in $\mathcal{D}^b\mathrm{Coh}_{\mathcal{P}^{(1)}}^{\mathbb{G}_{\mathbf{m}}}(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)})$ such that $\mathcal{F}_w \cong \zeta_{\mathcal{P}}(\mathcal{G}_w)$. Moreover, this object satisfies

$$(j_{\mathcal{P},\mathbb{G}_{\mathbf{m}}})_*(\rho_{\mathcal{P},\mathbb{G}_{\mathbf{m}}})^*\mathcal{G}_w \ \cong \ \mathcal{L}_{\tau_0 w}^{y,\mathrm{gr}} \otimes_{\widetilde{\mathcal{N}}^{(1)}} \mathcal{O}_{\widetilde{\mathcal{N}}^{(1)}}(\rho)\langle N_{\mathcal{P}} - N\rangle. \quad (10.2.10)$$

Consider now $\widetilde{\Upsilon}_\lambda^{\mathcal{P}}(\mathcal{G}_w)$. This is an object of $\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}(U_{\mathcal{P}}^\lambda)$. It follows from equation (10.2.10), Proposition 10.1.4 and equation (10.1.3) that its image under the composition

$$\mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg,gr}}(U_{\mathcal{P}}^\lambda) \xrightarrow{\mathrm{For}} \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}(U_{\mathcal{P}}^\lambda) \xrightarrow{(\phi_{\mathcal{P}}^\lambda)^*} \mathcal{D}^b\mathrm{Mod}_0^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})^\lambda)$$

is the simple module $L(\tau_0 w \bullet \lambda_0)$. Hence $\tau_0 w \in I_\lambda$, and a lift (hence all of them) of $L_{\mathcal{P}}(\tau_0 w \bullet \lambda_0)$ is in the essential image of $\widetilde{\Upsilon}_\lambda^{\mathcal{P}}$. If we set $L_{\mathcal{P}}^{\mathrm{gr}}(\tau_0 w \bullet \lambda_0) := \widetilde{\Upsilon}_\lambda^{\mathcal{P}}(\mathcal{G}_w)$ and $\mathcal{L}_{\mathcal{P},\tau_0 w}^{y,\mathrm{gr}} := \mathcal{G}_w$, then isomorphism (10.2.5) is clearly true in this case.

In particular, we have proved that $\tau_0 W_\mu^0 \subseteq I_\lambda$. As these two sets have the same cardinality (see equation (10.2.2)), we deduce that they coincide. This finishes the proof of Theorem 10.2.4. □

10.3 Koszulity of singular blocks of $(\mathcal{U}\mathfrak{g})_0$

The following theorem follows from Theorem 10.2.4, exactly as Theorem 9.5.1 follows from Theorem 4.4.3.

Theorem 10.3.1. *Let λ, μ be as in subsection 10.2, and keep the assumptions of Theorem 10.2.4. There exists a Koszul ring $B_{\mathcal{P}}$, which is naturally a $\Gamma(\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)}, \mathcal{O}_{\widetilde{\mathcal{N}}_{\mathcal{P}}^{(1)}})$-algebra, and equivalences of categories*

$$\begin{aligned} \mathrm{Mod}_0^{\mathrm{fg}}(B_{\mathcal{P}}) &\cong \mathrm{Mod}_0^{\mathrm{fg}}(U_{\mathcal{P}}^{\lambda}) \\ \mathrm{Mod}^{\mathrm{fg}}((B_{\mathcal{P}})^!) &\cong \mathrm{Mod}_{\mu}^{\mathrm{fg}}((\mathcal{U}\mathfrak{g})_0). \end{aligned}$$

In particular, the ring $(\mathcal{U}\mathfrak{g})_0^{\hat{\mu}}$ can be endowed with a Koszul grading.

For any $\nu \in \mathbb{X}$, there exists a weight μ in the orbit $W'_{\mathrm{aff}} \bullet \nu$, a standard parabolic subgroup P, and a weight λ which satisfy the hypotheses of Theorem 10.3.1 (see *e.g.* [BMR06, 1.5.2]). Hence the ring $(\mathcal{U}\mathfrak{g})_0^{\hat{\nu}} = (\mathcal{U}\mathfrak{g})_0^{\hat{\mu}}$ can be endowed with a Koszul grading for $p \gg 0$. As there are finitely many blocks, all the blocks of $(\mathcal{U}\mathfrak{g})_0$ can be endowed with a Koszul grading if $p \gg 0$. Finally, by [AJS94, F.4] (in fact the implication we use is trivial) we deduce:

Corollary 10.3.2. *For $p \gg 0$, the algebra $(\mathcal{U}\mathfrak{g})_0$ can be endowed with a Koszul grading.*

10.4 Remark on the choice of λ

Let $p > h$. Fix a parabolic subgroup $P \supset B$, and let $I \subset \Phi$ be the corresponding set of simple roots. In subsection 10.2, we have chosen λ such that the closure of its alcove contains a weight μ of singularity P, *i.e.* an integral weight in a facet which is open in $H_P := \{\nu \in \mathbb{X} \otimes_{\mathbb{Z}} \mathbb{R} \mid \forall \alpha \in I,\ \langle \nu + \rho, \alpha^\vee \rangle = 0\}$. It is not clear *a priori* that any regular weight $\lambda \in \mathbb{X}_{\mathcal{P}}$ satisfies this assumption. But it is indeed the case.

Let us check this fact. We can assume that G is quasi simple, *i.e.* R is irreducible. Let A_0 denote the fundamental alcove, and let $w \in W'_{\mathrm{aff}}$ be such that $A = w \bullet A_0$. What we have to prove is that $\overline{A} \cap H_P$ contains an integral weight in an open facet of H_P, or that $\overline{A_0} \cap (w^{-1} \bullet H_P)$ contains an integral weight in an open facet of $w^{-1} \bullet H_P$.

Write $w = t_\nu v$, with $\nu \in \mathbb{X}$ and $v \in W$. Let $\lambda_0 \in C_0$ be such that $\lambda = w \bullet \lambda_0$. If $\alpha \in I$ we have

$$0 = \langle \lambda, \alpha^\vee \rangle = \langle \lambda_0 + \rho, v^{-1}\alpha^\vee \rangle - 1 + p\langle \nu, \alpha^\vee \rangle.$$

By definition of C_0 we have $|\langle \lambda_0 + \rho, v^{-1}\alpha^\vee \rangle| < p$. Hence either (i) $\langle \nu, \alpha^\vee \rangle = 0$ and $\langle \lambda_0 + \rho, v^{-1}\alpha^\vee \rangle = 1$ (in this case $v^{-1}\alpha$ has to be a simple root), or (ii) $\langle \nu, \alpha^\vee \rangle = 1$ and $\langle \lambda_0 + \rho, v^{-1}\alpha^\vee \rangle = 1 - p$ (in this case $v^{-1}\alpha$ has to be the opposite of the highest short root). It follows that $\overline{A_0} \cap w^{-1} \bullet H_P$ is the closure of the facet of $\overline{A_0}$ defined by the simple roots appearing in (i) (if there are any) and the affine simple root (if case (ii) occurs). This facet contains integral weights because it is the image under w of an open facet in H_P. This concludes the proof of the claim.

Hence Theorem 10.3.1 gives a Koszul duality for *all* algebras $U_{\mathcal{P}}^{\lambda}$.

Chapter IV

Linear Koszul duality in a general setting

In this chapter we give a "linear Koszul duality" result, in the spirit of Theorem III.2.3.11, but in a more general context. Let us point out, however, that Theorem III.2.3.11 is *not* a particular case of the main result of this chapter. In particular, the equivalence we construct here is *contravariant*, while the equivalence of Theorem III.2.3.11 is *covariant*.

The setting we use for (quasi-)coherent dg-sheaves on dg-schemes is different from the one of chapter III. In particular in this chapter every sheaf on a scheme is quasi-coherent. Hence we will not write the superscript "qc" for the corresponding categories.

This chapter is a joint work with Ivan Mirković. It was prepublished in [MR08].

Introduction

0.1

Koszul duality is an algebraic formalism of Fourier transform which is often deep and mysterious in applications. For instance, Bezrukavnikov has noticed that it exchanges monodromy and the Chern class – the same as mirror duality, while the work of Beilinson, Ginzburg and Soergel ([BGS96]) has made Koszul duality an essential ingredient of representation theory.

The case of *linear Koszul duality* studied here has a simple geometric content which appears in a number of applications. For two vector subbundles F_1, F_2 of a vector bundle E (over a noetherian, integral, separated, regular base scheme), linear Koszul duality provides a (contravariant) equivalence of derived categories of $\mathbb{G}_{\mathbf{m}}$-equivariant coherent sheaves on the differential graded scheme $F_1 \overset{R}{\cap}_E F_2$ obtained as derived intersection of subbundles inside a vector bundle, and the corresponding object $F_1^{\perp} \overset{R}{\cap}_{E^*} F_2^{\perp}$ inside the dual vector bundle.

The origin of the linear duality observation is Kashiwara's isomorphism of Borel-Moore homology groups

$$\mathcal{H}_*(F_1 \cap_E F_2) \cong \mathcal{H}_*(F_1^\perp \cap_{E^*} F_2^\perp)$$

given by a Fourier transform for constructible sheaves. The Iwahori-Matsumoto involution for *graded* affine Hecke algebras has been realized as Kashiwara's Fourier isomorphism in equivariant Borel-Moore homology ([EM97]). The standard affine Hecke algebras have analogous realization in K-theory (the K-homology) and this suggested that Kashiwara's isomorphism lifts to K-homology, but natural isomorphisms of K-homology groups should come from equivalences of triangulated categories of coherent sheaves.

0.2

Let us describe more precisely the content of this chapter. We start in section 1 with generalities on sheaves on dg-schemes. In section 2 we construct the relevant Koszul type complexes, in section 3 we prove the equivalence of categories, and in section 4 we give the geometric interpretation of this duality. The idea is that the statement is a particular case of the standard Koszul duality in the generality of dg-vector bundles. However, because of convergence problems for spectral sequences, we are able to make sense of this duality only for the dg-vector bundles with at most 2 non-zero terms. Furthermore, the constructions would simplify if we were not interested in applications in positive characteristic, as in characterististic zero one could think of the Kosul complex of a vector bundle $\mathcal{V}$ as the symmetric algebra of the acyclic complex $\mathcal{V} \xrightarrow{\mathrm{Id}} \mathcal{V}$ (where the first term is in degree -1, and the second one in degree 0).

0.3

In a sequel to [MR08] we will show that the linear Koszul duality in K-homology is indeed a quantization of Kashiwara's Fourier isomorphism – the two are related by the Chern character. We will also verify that the linear Koszul duality in equivariant K-homology gives a geometric realization of the Iwahori-Matsumoto involution on (extended) affine Hecke algebras. This concerns one typical use of linear Koszul duality. Consider a partial flag variety $\mathcal{P}$ of a group G (either a reductive algebraic group in very good characteristic or a loop group[1]), and a subgroup K that acts on $\mathcal{P}$ with countably many orbits. Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras, choose E to be the trivial bundle $\mathcal{P} \times \mathfrak{g}^*$, F_1 the cotangent subbundle $T^*\mathcal{P}$ and $F_2 = \mathcal{P} \times \mathfrak{k}^\perp$. Now $F_1 \overset{R}{\cap}_E F_2$ is a differential graded version of the Lagrangian $\Lambda_K \subset T^*\mathcal{P}$, the union of all conormals to K-orbits in $\mathcal{P}$, and $F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp$ is the stabilizer dg-scheme for the action of the Lie algebra $\mathfrak{k}$ on $\mathcal{P}$. If K is the Borel subgroup then $F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp$ is homotopic to $F_1 \overset{R}{\cap}_E F_2$ and linear Koszul duality provides an involution on the K-group of equivariant coherent sheaves on Λ_K.

Let us conclude by proposing some further applications of linear Koszul duality. The above application to Iwahori-Matsumoto involutions should extend to its generalization,

[1]Let us point out that the application to loop groups would require an extension of our constructions to the case of infinite dimensional varieties, or ind-schemes, which is not proved here.

the Aubert involution on irreducible representations of p-adic groups ([Aub95]). Linear duality should be an ingredient in a geometric realization (proposed in [BFM05]) of the *Cherednik Fourier transform* (essentially an involution on the Cherednik Hecke algebra), in the Grojnowski-Garland realization of Cherednik Hecke algebras as equivariant K-groups of Steinberg varieties for affine flag varieties (see [Vas05]). The appearence of linear Koszul duality for conormals to Bruhat cells should be a classical limit of the Beilinson-Ginzburg-Soergel Koszul duality for the mixed category $\mathcal{O}$ ([BGS96]), as mixed Hodge modules come with a deformation (by Hodge filtration), to a coherent sheaf on the characteristic variety.

1 Generalities on sheaves of dg-algebras and dg-schemes

In this section X is any noetherian scheme satisfying the following assumption[2]:

$$(*) \qquad \begin{array}{c}\text{for any coherent sheaf } \mathcal{F} \text{ on } X, \text{ there exists a locally free} \\ \text{sheaf of finite rank } \mathcal{E} \text{ and a surjection } \mathcal{E} \twoheadrightarrow \mathcal{F}.\end{array}$$

We introduce basic definitions concerning dg-schemes and quasi-coherent dg-sheaves, mainly following [CFK01] and [Ric08b] (*i.e.* chapter III).

1.1 Definitions

Recall the definitions of sheaves of $\mathcal{O}_X$-dg-algebras and dg-modules given in III.1.1.

Definition 1.1.1. A dg-scheme is a pair $\mathbf{X} = (X, \mathcal{A})$ where X is a noetherian scheme satisfying $(*)$, and $\mathcal{A}$ is a non-positively graded, graded-commutative $\mathcal{O}_X$-dg-algebra such that $\mathcal{A}^i$ is a quasi-coherent $\mathcal{O}_X$-module for any $i \in \mathbb{Z}_{\leq 0}$.

Definition 1.1.2. Let $\mathbf{X} = (X, \mathcal{A})$ be a dg-scheme.

(i) A quasi-coherent dg-sheaf $\mathcal{F}$ on $\mathbf{X}$ is an $\mathcal{A}$-dg-module such that $\mathcal{F}^i$ is a quasi-coherent $\mathcal{O}_X$-module for any $i \in \mathbb{Z}$.

(ii) A coherent dg-sheaf $\mathcal{F}$ on $\mathbf{X}$ is a quasi-coherent dg-sheaf whose cohomology $H(\mathcal{F})$ is a locally finitely generated sheaf of $H(\mathcal{A})$-modules.

We denote by $\mathcal{C}(\mathbf{X})$, or $\mathcal{C}(X, \mathcal{A})$, the category of quasi-coherent dg-sheaves on the dg-scheme $\mathbf{X}$, and by $\mathcal{D}(\mathbf{X})$, or $\mathcal{D}(X, \mathcal{A})$, the associated derived category (*i.e.* the localization of the homotopy category of $\mathcal{C}(\mathbf{X})$ with respect to quasi-isomorphisms).

Similarly, we denote by $\mathcal{C}^{\mathrm{c}}(\mathbf{X})$ or $\mathcal{C}^{\mathrm{c}}(X, \mathcal{A})$, $\mathcal{D}^{\mathrm{c}}(\mathbf{X})$ or $\mathcal{D}^{\mathrm{c}}(X, \mathcal{A})$, the full subcategories whose objects are the coherent dg-sheaves.

If $\mathbf{X}$ is an ordinary scheme, *i.e.* if $\mathcal{A} = \mathcal{O}_X$, then we have equivalences

$$\mathcal{D}(\mathbf{X}) \cong \mathcal{D}\mathrm{QCoh}(X), \quad \mathcal{D}^{\mathrm{c}}(\mathbf{X}) \cong \mathcal{D}^b\mathrm{Coh}(X).$$

Let us stress that these definitions and notations are *different* from the ones used in chapter III or [Ric08b] (in *loc. cit.*, we only require the cohomology of $\mathcal{F}$ to be quasi-coherent). This definition will be more suited to our purposes here. Moreover, these

[2]See *e.g.* the remarks before [CFK01, Lemma 2.3.4] for comments on this assumption.

two definitions coincide under reasonable assumptions. For the categories of coherent dg-sheaves in all the cases we consider here, this can be deduced from Proposition III.3.2.4.

1.2 K-flat resolutions

Let us fix a dg-scheme $\mathbf{X} = (X, \mathcal{A})$. If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{A}$-dg-modules, we define as usual the tensor product $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ (see III.1.2). It has a natural structure of an $\mathcal{A}$-dg-module (here $\mathcal{A}$ is graded-commutative, hence we do not have to distinguish between left and right dg-modules, see III.1.1).

Recall the definition of a K-flat dg-module (see [Spa88]):

Definition 1.2.1. An $\mathcal{A}$-dg-module $\mathcal{F}$ is said to be *K-flat* if for every $\mathcal{A}$-dg-module $\mathcal{G}$ such that $H(\mathcal{G}) = 0$, we have $H(\mathcal{G} \otimes_{\mathcal{A}} \mathcal{F}) = 0$.

Using [Spa88, 3.4, 5.4.(c)] and assumption $(*)$, one easily proves the following lemma.

Lemma 1.2.2. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-dg-module. There exist a quasi-coherent, K-flat $\mathcal{O}_X$-dg-module $\mathcal{P}$ and a surjective quasi-isomorphism $\mathcal{P} \xrightarrow{\mathrm{qis}} \mathcal{F}$.*

Then, using the induction functor $\mathcal{F} \mapsto \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F}$, the following proposition can be proved exactly as Theorem III.1.3.5.

Proposition 1.2.3. *Let $\mathcal{F}$ be a quasi-coherent dg-sheaf on $\mathbf{X}$. There exist a quasi-coherent dg-sheaf $\mathcal{P}$ on $\mathbf{X}$, K-flat as an $\mathcal{A}$-dg-module, and a quasi-isomorphism $\mathcal{P} \xrightarrow{\mathrm{qis}} \mathcal{F}$.*

1.3 Invariance under quasi-isomorphisms

In this subsection we prove that the categories $\mathcal{D}(\mathbf{X})$, $\mathcal{D}^{\mathrm{c}}(\mathbf{X})$ depend on $\mathcal{A}$ only up to quasi-isomorphism.

Let X be a noetherian scheme satisfying $(*)$, and let $\mathbf{X} = (X, \mathcal{A})$ and $\mathbf{X}' = (X, \mathcal{B})$ be two dg-schemes with base scheme X. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a morphism of sheaves of $\mathcal{O}_X$-dg-algebras. There is a natural functor

$$\phi^* : \mathcal{C}(\mathbf{X}') \to \mathcal{C}(\mathbf{X})$$

(restriction of scalars), which induces a functor

$$R\phi^* : \mathcal{D}(\mathbf{X}') \to \mathcal{D}(\mathbf{X}).$$

Similarly, there is a natural functor

$$\phi_* : \left\{ \begin{array}{ccc} \mathcal{C}(\mathbf{X}) & \to & \mathcal{C}(\mathbf{X}') \\ \mathcal{F} & \mapsto & \mathcal{B} \otimes_{\mathcal{A}} \mathcal{F} \end{array} \right. .$$

We refer to [Del73] or [Kel96] for generalities on localization of triangulated categories and derived functors (in the sense of Deligne). The following lemma is borrowed from

[Spa88, 5.7] (see also Lemma III.1.3.6), and implies that K-flat $\mathcal{A}$-dg-modules are split on the left for the functor ϕ_*. Using Proposition 1.2.3, it follows that ϕ_* admits a left derived functor

$$L\phi_* : \mathcal{D}(\mathbf{X}) \to \mathcal{D}(\mathbf{X}').$$

Lemma 1.3.1. *Let $\mathcal{F}$ be an object of $\mathcal{C}(X, \mathcal{A})$ which is acyclic (*i.e. *$H(\mathcal{F}) = 0$) and K-flat as an $\mathcal{A}$-dg-module. Then $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{F}$ is acyclic.*

The following result is an immediate extension of [BL94, 10.12.5.1] (see also Proposition III.1.5.6).

Proposition 1.3.2. (i) *Assume $\phi : \mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism. Then the functors $L\phi_*$, $R\phi^*$ are quasi-inverse equivalences of categories*

$$\mathcal{D}(\mathbf{X}) \cong \mathcal{D}(\mathbf{X}').$$

(ii) *These equivalences restrict to equivalences*

$$\mathcal{D}^c(\mathbf{X}) \cong \mathcal{D}^c(\mathbf{X}').$$

Proof: Statement (i) can be proved as in [BL94, 10.12.5.1] or Proposition III.1.5.6. Then, clearly, for $\mathcal{G}$ in $\mathcal{D}(\mathbf{X}')$ we have $\mathcal{G} \in \mathcal{D}^c(\mathbf{X}')$ iff $R\phi^*\mathcal{G} \in \mathcal{D}^c(\mathbf{X})$. Point (ii) follows. □

1.4 Derived intersection

Using Proposition 1.3.2, one can consider dg-schemes "up to quasi-isomorphism", *i.e.* identify the dg-schemes $(X, \mathcal{A})$ and $(X, \mathcal{B})$ whenever $\mathcal{A}$ and $\mathcal{B}$ are quasi-isomorphic.

As a typical example, we define the derived intersection of two closed subschemes. Consider a scheme X, and two closed subschemes Y and Z. Let us denote by $i : Y \hookrightarrow X$ and $j : Z \hookrightarrow X$ the closed embeddings. Consider the sheaf of dg-algebras $i_*\mathcal{O}_Y \overset{L}{\otimes}_{\mathcal{O}_X} j_*\mathcal{O}_Z$ on X. It is well defined up to quasi-isomorphism: if $\mathcal{A}_Y \to i_*\mathcal{O}_Y$, respectively $\mathcal{A}_Z \to j_*\mathcal{O}_Z$ are quasi-isomorphisms of non-positively graded, graded-commutative sheaves of $\mathcal{O}_X$-dg-algebras[3], with $\mathcal{A}_Y$ and $\mathcal{A}_Z$ quasi-coherent and K-flat over $\mathcal{O}_X$, then $i_*\mathcal{O}_Y \overset{L}{\otimes}_{\mathcal{O}_X} j_*\mathcal{O}_Z$ is quasi-isomorphic to $\mathcal{A}_Y \otimes_{\mathcal{O}_X} j_*\mathcal{O}_Z$, or to $i_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{A}_Z$, or to $\mathcal{A}_Y \otimes_{\mathcal{O}_X} \mathcal{A}_Z$.

Definition 1.4.1. The right derived intersection of Y and Z in X is the dg-scheme

$$Y \overset{R}{\cap}_X Z := (X,\ i_*\mathcal{O}_Y \overset{L}{\otimes}_{\mathcal{O}_X} j_*\mathcal{O}_Z),$$

defined up to quasi-isomorphism.

To be really precise, only the derived categories $\mathcal{D}(Y \overset{R}{\cap}_X Z)$, $\mathcal{D}^c(Y \overset{R}{\cap}_X Z)$ are well defined (up to equivalence). This is all we will use here.

[3] See *e.g.* [CFK01, 2.6.1] for a proof of the existence of such resolutions.

2 Generalized Koszul complexes

In this section we introduce the dg-algebras we are interested in, and define our Koszul complexes.

2.1 Notation and definitions

From now on X is a noetherian, integral, separated, regular scheme of dimension d. Observe that X satisfies condition $(*)$ by [Har77, III.Ex.6.8]. We will consider $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebras on X, *i.e.* sheaves of $\mathcal{O}_X$-algebras $\mathcal{A}$, endowed with a $\mathbb{Z}^2$-grading

$$\mathcal{A} = \bigoplus_{i,j \in \mathbb{Z}} \mathcal{A}^i_j$$

and an $\mathcal{O}_X$-linear differential $d_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$, of bidegree $(1,0)$, *i.e.* such that $d_{\mathcal{A}}(\mathcal{A}^i_j) \subseteq \mathcal{A}^{i+1}_j$, and satisfying

$$d_{\mathcal{A}}(a \cdot b) = d_{\mathcal{A}}(a) \cdot b + (-1)^i a \cdot d_{\mathcal{A}}(b)$$

for $a \in \mathcal{A}^i_j$, $b \in \mathcal{A}$. The basic example is $\mathcal{O}_X$, endowed with the trivial grading (*i.e.* it is concentrated in bidegree $(0,0)$) and the trivial differential.

A $\mathbb{G}_{\mathbf{m}}$-equivariant dg-module over $\mathcal{A}$ is a sheaf $\mathcal{M}$ of $\mathbb{Z}^2$-graded $\mathcal{A}$-modules endowed with a differential $d_{\mathcal{M}}$ of bidegree $(1,0)$ satisfying

$$d_{\mathcal{M}}(a \cdot m) = d_{\mathcal{A}}(a) \cdot m + (-1)^i a \cdot d_{\mathcal{M}}(m)$$

for $a \in \mathcal{A}^i_j$, $m \in \mathcal{M}$.

We will only consider quasi-coherent ($\mathbb{G}_{\mathbf{m}}$-equivariant) $\mathcal{O}_X$-dg-algebras. If $\mathcal{A}$ is such a dg-algebra, we denote by $\mathcal{C}_{\mathrm{gr}}(\mathcal{A})$ the category of quasi-coherent $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{A}$-dg-modules, *i.e.* $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{A}$-dg-modules $\mathcal{M}$ such that $\mathcal{M}^i_j$ is $\mathcal{O}_X$-quasi-coherent for any indices i, j.

If $\mathcal{M}$ is a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{A}$-dg-module, and m is a local section of $\mathcal{M}^i_j$, we write $|m| = i$. This integer is called the *cohomological degree* of m, while j is called its *internal degree*. We can define two shifts in $\mathcal{C}_{\mathrm{gr}}(\mathcal{A})$: $[n]$, shifting the cohomological degree, and $\langle m \rangle$, shifting the internal degree. More precisely we set

$$(\mathcal{M}[n]\langle m \rangle)^i_j = \mathcal{M}^{i+n}_{j-m}.$$

Beware that in our conventions $\langle 1 \rangle$ is a "homological" shift, *i.e.* it shifts the internal degrees to the right. Also, we use the same conventions as in [BL94, §10] or III.1.1 concerning the shift $[1]$, *i.e.* the differential of $\mathcal{M}[1]$ is opposite to the differential of $\mathcal{M}$.

If $\mathcal{M}$ and $\mathcal{N}$ are two $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-modules, there is a natural structure of $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module on the tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$, with differential defined on homogeneous local sections by

$$d_{\mathcal{M}\otimes\mathcal{N}}(m \otimes n) = d_{\mathcal{M}}(m) \otimes n + (-1)^{|m|} m \otimes d_{\mathcal{N}}(n).$$

If $\mathcal{M}$ is a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module, we define the $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module $\mathcal{M}^\vee$ as the graded dual of $\mathcal{M}$, *i.e.* the dg-module with (i,j)-component

$$(\mathcal{M}^\vee)^i_j := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}^{-i}_{-j}, \mathcal{O}_X),$$

and with differential defined by $d_{\mathcal{M}^\vee}(f) = -(-1)^{|f|} f \circ d_{\mathcal{M}}$ for $f \in \mathcal{M}^\vee$ homogeneous. If $\mathcal{M}$ and $\mathcal{N}$ are two $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-modules, there is a natural morphism defined (on homogeneous local sections) by

$$\begin{cases} \mathcal{M}^\vee \otimes_{\mathcal{O}_X} \mathcal{N}^\vee & \to & (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})^\vee \\ f \otimes g & \mapsto & \big(m \otimes n \mapsto (-1)^{|m|\cdot|g|} f(m) \cdot g(n)\big) \end{cases}, \tag{2.1.1}$$

which is an isomorphism *e.g.* if the homogeneous components of $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ are locally free of finite rank. If $\mathcal{M}$ is a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module such that $\mathcal{M}^i_j$ is locally-free of finite rank for any i,j, then there is an isomorphism

$$\begin{cases} \mathcal{M} & \xrightarrow{\sim} & (\mathcal{M}^\vee)^\vee \\ m & \mapsto & \big(f \mapsto (-1)^{|f|\cdot|m|} f(m)\big) \end{cases}. \tag{2.1.2}$$

Let us recall the definition of the truncation functors. If $\mathcal{M}$ is a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module and if $n \in \mathbb{Z}$, we define the $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module $\tau_{\geq n}(\mathcal{M})$ by

$$\tau_{\geq n}(\mathcal{M})^i_j := \begin{cases} 0 & \text{if } i < n \\ \mathcal{M}^n_j / d_{\mathcal{M}}(\mathcal{M}^{n-1}_j) & \text{if } i = n \\ \mathcal{M}^i_j & \text{if } i > n \end{cases},$$

with the differential induced by $d_{\mathcal{M}}$. There is a natural morphism $\mathcal{M} \to \tau_{\geq n}(\mathcal{M})$. Similarly, we define the $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module $\tau_{\leq n}(\mathcal{M})$ by

$$\tau_{\leq n}(\mathcal{M}) := \mathrm{Ker}\big(\mathcal{M} \to \tau_{\geq n+1}(\mathcal{M})\big).$$

Observe that if $\mathcal{A}$ is a $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebra with $\mathcal{A}^i_j = 0$ for $i > 0$, and if $\mathcal{M}$ is a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{A}$-dg-module, then $\tau_{\geq n}(\mathcal{M})$ and $\tau_{\leq n}(\mathcal{M})$ are again $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{A}$-dg-modules.

If $\mathcal{M}$ is a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module, we denote by $\mathrm{Sym}(\mathcal{M})$ the graded-symmetric algebra of $\mathcal{M}$ over $\mathcal{O}_X$ (*i.e.* the quotient of the tensor algebra of $\mathcal{M}$ by the relations $m \otimes n = (-1)^{|m|\cdot|n|} n \otimes m$), considered as a $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebra with differential induced by $d_{\mathcal{M}}$. Similarly, if $\mathcal{F}$ is any $\mathcal{O}_X$-module, we denote by $\mathrm{S}_{\mathcal{O}_X}(\mathcal{F})$, or simply $\mathrm{S}(\mathcal{F})$, the symmetric algebra of $\mathcal{F}$.

Let us consider two locally free sheaves of finite rank $\mathcal{V}$ and $\mathcal{W}$ on X, and a morphism of sheaves $f : \mathcal{V} \to \mathcal{W}$. Let $\mathcal{V}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{V}, \mathcal{O}_X)$ and $\mathcal{W}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{W}, \mathcal{O}_X)$ be the dual locally free sheaves, and $f^\vee : \mathcal{W}^\vee \to \mathcal{V}^\vee$ be the morphism induced by f. Let us consider the $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-modules (or complexes of graded $\mathcal{O}_X$-modules)

$$\mathcal{X} := \big(\cdots \to 0 \to \mathcal{V} \xrightarrow{f} \mathcal{W} \to 0 \to \cdots\big),$$

where $\mathcal{V}$ is in bidegree $(-1,2)$ and $\mathcal{W}$ is in bidegree $(0,2)$, and

$$\mathcal{Y} := \big(\cdots \to 0 \to \mathcal{W}^\vee \xrightarrow{-f^\vee} \mathcal{V}^\vee \to 0 \to \cdots \big),$$

where $\mathcal{W}^\vee$ is in bidegree $(-1,-2)$ and $\mathcal{V}^\vee$ is in bidegree $(0,-2)$.

In sections 2 and 3 we will consider the following $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebras:

$$\begin{array}{rcl} \mathcal{T} & := & \mathrm{Sym}(\mathcal{X}), \\ \mathcal{R} & := & \mathrm{Sym}(\mathcal{Y}), \\ \mathcal{S} & := & \mathrm{Sym}(\mathcal{Y}[-2]). \end{array}$$

For example, the generators of $\mathcal{T}$ are in bidegrees $(-1,2)$ and $(0,2)$, and the generators of $\mathcal{S}$ are in bidegrees $(1,-2)$ and $(2,-2)$.

If $\mathcal{M}$ is a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{S}$-dg-module, the dual $\mathcal{M}^\vee$ has a natural structure of a $\mathcal{S}$-dg-module, constructed as follows. The grading and the differential are defined as above, and the $\mathcal{S}$-action is defined by

$$(s \cdot f)(m) = (-1)^{|s| \cdot |f|} f(s \cdot m),$$

for homogeneous local sections s of $\mathcal{S}$ and f of $\mathcal{M}^\vee$.

If $\mathcal{N}$ is a $\mathcal{T}$-dg-module, respectively a $\mathcal{R}$-dg-module, the same formulas define on $\mathcal{N}^\vee$ a structure of a $\mathcal{T}$-dg-module, respectively a $\mathcal{R}$-dg-module.

2.2 Reminder on the spectral sequence of a double complex

Let us recall a few facts on the spectral sequence of a double complex. Let $(C^{p,q})_{p,q\in\mathbb{Z}}$ be a double complex (in any abelian category), with differentials d' (of bidegree $(1,0)$) and d'' (of bidegree $(0,1)$). Let $\mathrm{Tot}(C)$ be the total complex of C, *i.e.* the complex with n-term

$$\mathrm{Tot}(C)^n = \bigoplus_{p+q=n} C^{p,q},$$

and with differential $d' + d''$. The following result is proved *e.g.* in [God64, I.4].

Proposition 2.2.1. *Assume one of the following conditions is satisfied:*

1. *There exists $N \in \mathbb{Z}$ such that $C^{p,q} = 0$ for $p > N$.*

2. *There exists $N \in \mathbb{Z}$ such that $C^{p,q} = 0$ for $q < N$.*

Then there is a converging spectral sequence

$$E_1^{p,q} = H^q(C^{p,*}, d'') \Rightarrow H^{p+q}(\mathrm{Tot}(C)).$$

2.3 Reminder on Koszul complexes

Let A be a commutative ring, and V be a free A-module of finite rank. Let $V^\vee = \mathrm{Hom}_A(V, A)$ be the dual A-module, and consider the natural morphism

$$i : A \to \mathrm{Hom}_A(V, V) \cong V^\vee \otimes_A V,$$

sending 1_A to Id_V. Let us first consider the bigraded algebras $\Lambda(V[-1]\langle -2\rangle)$, the exterior algebra of V placed in bidegree $(1, -2)$, and $\mathrm{S}(V^\vee\langle 2\rangle)$, the symmetric algebra of $V^\vee$ placed in bidegree $(0, 2)$. The algebra $\Lambda(V[-1]\langle -2\rangle)$ acts on the dual $(\Lambda(V[-1]\langle -2\rangle))^\vee$ via

$$(t \cdot f)(s) = (-1)^{|t|\cdot|f|} f(ts),$$

where t, s are homogeneous elements of $\Lambda(V[-1]\langle -2\rangle)$, and f is an homogeneous element of $(\Lambda(V[-1]\langle -2\rangle))^\vee$.

Consider the usual Koszul complex

$$\mathrm{Koszul}_1(V) := \mathrm{S}(V^\vee\langle 2\rangle) \otimes_A (\Lambda(V[-1]\langle -2\rangle))^\vee, \tag{2.3.1}$$

where the differential is the composition of the morphism

$$\begin{cases} \mathrm{S}(V^\vee) \otimes_A (\Lambda(V))^\vee & \to & \mathrm{S}(V^\vee) \otimes_A (\Lambda(V))^\vee \\ s \otimes t & \mapsto & (-1)^{|s|} s \otimes t \end{cases}$$

followed by the morphism induced by i

$$\mathrm{S}(V^\vee) \otimes_A (\Lambda(V))^\vee \to \mathrm{S}(V^\vee) \otimes_A V^\vee \otimes_A V \otimes_A (\Lambda(V))^\vee$$

and finally followed by the morphism

$$\mathrm{S}(V^\vee) \otimes_A V^\vee \otimes_A V \otimes_A (\Lambda(V))^\vee \to \mathrm{S}(V^\vee) \otimes_A (\Lambda(V))^\vee$$

induced by the action of $V^\vee \subset \mathrm{S}(V^\vee)$ on $\mathrm{S}(V^\vee)$ by right multiplication and the action of $V \subset \Lambda(V)$ on $(\Lambda(V))^\vee$ described above. It is well-known (see *e.g.* [BGG78], [BGS96]) that this complex has cohomology only in degree 0, and more precisely that

$$H(\mathrm{Koszul}_1(V)) = A.$$

The complex $\mathrm{Koszul}_1(V)$ is a bounded complex of projective graded A-modules (here we consider A as a graded ring concentrated in degree 0). We can take its dual

$$\mathrm{Koszul}_2(V) := (\mathrm{Koszul}_1(V))^\vee \cong \Lambda(V[-1]\langle -2\rangle) \otimes_A (\mathrm{S}(V^\vee\langle 2\rangle))^\vee. \tag{2.3.2}$$

Again we have

$$H(\mathrm{Koszul}_2(V)) = A.$$

Now, let us consider the bigraded algebras $\Lambda(V[1]\langle -2\rangle)$, with generators in bidegree $(-1, -2)$, and $\mathrm{S}(V[-2]\langle 2\rangle)$, with generators in bidegree $(2, 2)$. We have a third Koszul complex

$$\mathrm{Koszul}_3(V) := \mathrm{S}(V^\vee[-2]\langle 2\rangle) \otimes_A (\Lambda(V[1]\langle -2\rangle))^\vee, \tag{2.3.3}$$

which may be defined as the bigraded module whose (i,j)-component is $(\mathrm{Koszul}_3(V))^i_j := (\mathrm{Koszul}_1(V))^{i-j}_j$, and with differential induced by that of $\mathrm{Koszul}_1(V)$. As above we have

$$H(\mathrm{Koszul}_3(V)) = A.$$

We can finally play the same game with the complex $\mathrm{Koszul}_2(V)$ and obtain the complex

$$\mathrm{Koszul}_4(V) \cong \Lambda(V[1]\langle -2\rangle) \otimes_A (\mathrm{S}(V^\vee[-2]\langle 2\rangle))^\vee \tag{2.3.4}$$

defined by $(\mathrm{Koszul}_4(V))^i_j = (\mathrm{Koszul}_2(V))^{i-j}_j$. Again we have

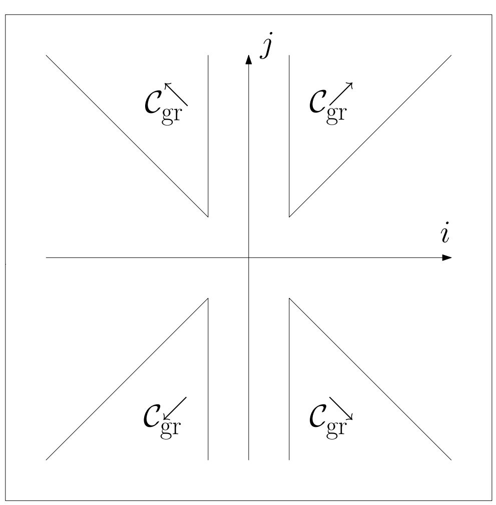
$$H(\mathrm{Koszul}_4(V)) = A.$$

2.4 Two functors

For any quasi-coherent $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebra $\mathcal{A}$ we define the category $\mathcal{C}_{\mathrm{gr}}^{\searrow}(\mathcal{A})$ of $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{A}$-dg-modules $\mathcal{M}$ such that $\mathcal{M}^i_j$ is a coherent $\mathcal{O}_X$-module for any indices i, j, and such that there exist integers N_1, N_2 such that $\mathcal{M}^i_j = 0$ for $i \leq N_1$ or $i + j \geq N_2$. Here the symbol "$\searrow$" indicates the region in the plane with coordinates (i,j) where the components $\mathcal{M}^i_j$ can be non-zero, as shown in the figure below.

Similarly, we define the categories $\mathcal{C}_{\mathrm{gr}}^{\swarrow}(\mathcal{A})$, $\mathcal{C}_{\mathrm{gr}}^{\nearrow}(\mathcal{A})$, $\mathcal{C}_{\mathrm{gr}}^{\nwarrow}(\mathcal{A})$ of $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{A}$-dg-modules $\mathcal{M}$ such that the $\mathcal{M}^i_j$'s are coherent and satisfy the following conditions:

$$\begin{array}{ll} \mathcal{C}_{\mathrm{gr}}^{\swarrow}(\mathcal{A}): & \mathcal{M}^i_j = 0 \text{ if } i \gg 0 \text{ or } i - j \ll 0, \\ \mathcal{C}_{\mathrm{gr}}^{\nearrow}(\mathcal{A}): & \mathcal{M}^i_j = 0 \text{ if } i \ll 0 \text{ or } i - j \gg 0, \\ \mathcal{C}_{\mathrm{gr}}^{\nwarrow}(\mathcal{A}): & \mathcal{M}^i_j = 0 \text{ if } i \gg 0 \text{ or } i + j \ll 0. \end{array}$$



In this subsection we define two *contravariant* functors

$$\mathscr{A} : \mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S}) \to \mathcal{C}^{\nwarrow}_{\mathrm{gr}}(\mathcal{T}), \quad \mathscr{B} : \mathcal{C}^{\nwarrow}_{\mathrm{gr}}(\mathcal{T}) \to \mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S}).$$

First, let us construct $\mathscr{A}$. If $\mathcal{M}$ is a $\mathcal{S}$-dg-module, we have defined in 2.1 the $\mathcal{S}$-dg-module $\mathcal{M}^{\vee}$. Let $\mathcal{M} \in \mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S})$. As a bigraded $\mathcal{O}_X$-module we set

$$\mathscr{A}(\mathcal{M}) = \mathcal{T} \otimes_{\mathcal{O}_X} \mathcal{M}^{\vee},$$

endowed with a $\mathcal{T}$-action by left multiplication on the first factor. The differential on $\mathscr{A}(\mathcal{M})$ is the sum of four terms. The first one is $d_1 := d_{\mathcal{T}} \otimes \mathrm{Id}_{\mathcal{M}^{\vee}}$, and the second one is $d_2 := \mathrm{Id}_{\mathcal{T}} \otimes d_{\mathcal{M}^{\vee}}$. Here the tensor product is taken in the graded sense, *i.e.* for homogeneous local sections t and f of $\mathcal{T}$ and $\mathcal{M}^{\vee}$ respectively we have $d_2(t \otimes f) = (-1)^{|t|} t \otimes d_{\mathcal{M}^{\vee}}(f)$. The third and fourth terms are "Koszul-type" differentials. Consider first the natural morphism $i : \mathcal{O}_X \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{V}) \cong \mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{V}^{\vee}$. Then d_3 is the composition of

$$\begin{cases} \mathcal{T} \otimes_{\mathcal{O}_X} \mathcal{M}^{\vee} & \to \quad \mathcal{T} \otimes_{\mathcal{O}_X} \mathcal{M}^{\vee} \\ \quad t \otimes f & \mapsto \quad (-1)^{|t|} t \otimes f \end{cases}$$

followed by the morphism induced by i

$$\mathcal{T} \otimes_{\mathcal{O}_X} \mathcal{M}^{\vee} \to \mathcal{T} \otimes_{\mathcal{O}_X} \mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{V}^{\vee} \otimes_{\mathcal{O}_X} \mathcal{M}^{\vee}$$

and finally followed by the morphism

$$\mathcal{T} \otimes_{\mathcal{O}_X} \mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{V}^{\vee} \otimes_{\mathcal{O}_X} \mathcal{M}^{\vee} \to \mathcal{T} \otimes_{\mathcal{O}_X} \mathcal{M}^{\vee}$$

induced by the right multiplication of $\mathcal{V} \subset \mathcal{T}$ on $\mathcal{T}$, and the left action of $\mathcal{V}^{\vee} \subset \mathcal{S}$ on $\mathcal{M}^{\vee}$. The differential d_4 is defined entirely similarly, replacing $\mathcal{V}$ by $\mathcal{W}$.

Let us choose a point $x \in X$. Then $\mathcal{V}_x$, $\mathcal{W}_x$ are free $\mathcal{O}_{X,x}$-modules of finite rank. Let $\{v_{\alpha}\}$ be a basis of $\mathcal{V}_x$, and $\{w_{\beta}\}$ be a basis of $\mathcal{W}_x$. Let $\{v_{\alpha}^*\}$, $\{w_{\beta}^*\}$ be the dual bases of $(\mathcal{V}^{\vee})_x$ and $(\mathcal{W}^{\vee})_x$, respectively. Then the morphism induced by d_3+d_4 on $\mathcal{T}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{M}^{\vee})_x$ can be written

$$(d_3 + d_4)(t \otimes f) = (-1)^{|t|} \big(\sum_{\alpha} t v_{\alpha} \otimes v_{\alpha}^* \cdot f + \sum_{\beta} t w_{\beta} \otimes w_{\beta}^* \cdot f \big) \tag{2.4.1}$$

for homogeneous local sections t of $\mathcal{T}$ and f of $\mathcal{M}^{\vee}$.

Using formula (2.4.1), one easily checks the relations

$$(d_1 + d_2)^2 = 0, \quad (d_3 + d_4)^2 = 0. \tag{2.4.2}$$

Further calculations prove the following formula:

$$(d_1 + d_2) \circ (d_3 + d_4) + (d_3 + d_4) \circ (d_1 + d_2) = 0. \tag{2.4.3}$$

It follows from formulas (2.4.2) and (2.4.3) that $d_{\mathscr{A}(\mathcal{M})} := d_1 + d_2 + d_3 + d_4$ is indeed a differential. Finally, one easily checks that $\mathscr{A}(\mathcal{M})$ is a $\mathcal{T}$-dg-module, and that it is an object of the category $\mathcal{C}^{\nwarrow}_{\mathrm{gr}}(\mathcal{T})$. Hence the (contravariant) functor

$$\mathscr{A} : \mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S}) \to \mathcal{C}^{\nwarrow}_{\mathrm{gr}}(\mathcal{T})$$

is well defined.

Now we define a functor $\mathscr{B}$ in the reverse direction, using similar formulas. Namely if $\mathcal{N}$ is a $\mathcal{T}$-dg-module, we have defined above the $\mathcal{T}$-dg-module $\mathcal{N}^\vee$. If $\mathcal{N} \in \mathcal{C}^{\nwarrow}_{\mathrm{gr}}(\mathcal{T})$, as a bigraded $\mathcal{O}_X$-module, we set

$$\mathscr{B}(\mathcal{N}) = \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{N}^\vee,$$

and we endow it with the $\mathcal{S}$-action by left multiplication on the first factor. The differential is again a sum of four terms. The first two are $d_1 := d_{\mathcal{S}} \otimes \mathrm{Id}_{\mathcal{N}^\vee}$ and $d_2 := \mathrm{Id}_{\mathcal{S}} \otimes d_{\mathcal{N}^\vee}$. The third one, denoted d_3, is defined as above as the composition of

$$\begin{cases} \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{N}^\vee & \to & \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{N}^\vee \\ s \otimes g & \mapsto & (-1)^{|s|} s \otimes g \end{cases}$$

followed by the morphism induced by $i' : \mathcal{O}_X \to \mathcal{V}^\vee \otimes_{\mathcal{O}_X} \mathcal{V}$

$$\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{N}^\vee \to \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{V}^\vee \otimes_{\mathcal{O}_X} \mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{N}^\vee$$

and finally followed by the morphism

$$\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{V}^\vee \otimes_{\mathcal{O}_X} \mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{N}^\vee \to \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{N}^\vee$$

induced by the right multiplication of $\mathcal{V}^\vee \subset \mathcal{S}$ on $\mathcal{S}$, and the left action of $\mathcal{V} \subset \mathcal{T}$ on $\mathcal{N}^\vee$. The differential d_4 is defined similarly, replacing $\mathcal{V}$ by $\mathcal{W}$. As above, one checks that $d_{\mathscr{B}(\mathcal{N})} := d_1 + d_2 + d_3 + d_4$ is a differential, which turns $\mathscr{B}(\mathcal{N})$ into a $\mathcal{S}$-dg-module, and even an object of $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S})$. For this final claim we use the fact that if $\mathcal{S}^k_l \neq 0$, then $k + l \leq 0$. As above, this proves that the (contravariant) functor

$$\mathscr{B} : \mathcal{C}^{\nwarrow}_{\mathrm{gr}}(\mathcal{T}) \to \mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S})$$

is well defined.

2.5 First generalized Koszul complex

Consider the object

$$\mathcal{K}^{(1)} := \mathscr{B}(\mathcal{T}) \in \mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S}).$$

It is concentrated in non-negative cohomological degrees, and in non-positive internal degrees.

Lemma 2.5.1. *The natural morphism $\mathcal{K}^{(1)} \to \mathcal{O}_X$ (projection on the $(0,0)$-component) is a quasi-isomorphism of $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{S}$-dg-modules.*

Proof. It is sufficient to prove that the localization of this morphism at any $x \in X$ is a quasi-isomorphism. We have isomorphisms

$$\begin{aligned} (\mathcal{K}^{(1)})_x &\cong (\mathcal{S}_x) \otimes_{\mathcal{O}_{X,x}} \mathcal{T}_x^\vee \\ &\cong \bigoplus_{i,j,k,l} \Lambda^i(\mathcal{W}_x^\vee) \otimes_{\mathcal{O}_{X,x}} \mathrm{S}^j(\mathcal{V}_x^\vee) \otimes_{\mathcal{O}_{X,x}} (\Lambda^k(\mathcal{V}_x))^\vee \otimes_{\mathcal{O}_{X,x}} (\mathrm{S}^l(\mathcal{W}_x))^\vee, \end{aligned}$$

where the symbol "$\vee$" denotes the dual $\mathcal{O}_{X,x}$-module, and where the term $\Lambda^i(\mathcal{W}_x^\vee) \otimes_{\mathcal{O}_{X,x}} \mathrm{S}^j(\mathcal{V}_x^\vee) \otimes_{\mathcal{O}_{X,x}} (\Lambda^k(\mathcal{V}_x))^\vee \otimes_{\mathcal{O}_{X,x}} (\mathrm{S}^l(\mathcal{W}_x))^\vee$ is in cohomological degree $i+2j+k$. The differential on $(\mathcal{K}^{(1)})_x$ is the sum of four terms: d_1, induced by the differential of $\mathcal{S}_x$; d_2, induced by the differential of $\mathcal{T}_x^\vee$; and d_3 and d_4, the Koszul differentials. The effect of these terms on the indices i, j, k, l may be described as follows:

$$d_1 : \left\{ \begin{array}{ccc} i & \mapsto & i-1 \\ j & \mapsto & j+1 \end{array} \right. , \ d_2 : \left\{ \begin{array}{ccc} k & \mapsto & k+1 \\ l & \mapsto & l-1 \end{array} \right. , \ d_3 : \left\{ \begin{array}{ccc} j & \mapsto & j+1 \\ k & \mapsto & k-1 \end{array} \right. , \ d_4 \left\{ \begin{array}{ccc} i & \mapsto & i+1 \\ l & \mapsto & l-1 \end{array} \right. .$$

Disregarding the internal grading, $(\mathcal{K}^{(1)})_x$ is the total complex of the double complex $(C^{p,q})_{p,q\in\mathbb{Z}}$ whose (p,q)-term is

$$C^{p,q} := \bigoplus_{\substack{p=j+k, \\ q=i+j}} \Lambda^i(\mathcal{W}_x^\vee) \otimes_{\mathcal{O}_{X,x}} \mathrm{S}^j(\mathcal{V}_x^\vee) \otimes_{\mathcal{O}_{X,x}} (\Lambda^k(\mathcal{V}_x))^\vee \otimes_{\mathcal{O}_{X,x}} (\mathrm{S}^l(\mathcal{W}_x))^\vee,$$

and whose differentials are $d' = d_1 + d_2$, $d'' = d_3 + d_4$. We have $C^{p,q} = 0$ if $q < 0$, hence by Proposition 2.2.1 there is a converging spectral sequence

$$E_1^{p,q} = H^q(C^{p,*}, d'') \Rightarrow H^{p+q}((\mathcal{K}^{(1)})_x).$$

It follows that, to prove the lemma, we only have to prove that the cohomology of $\mathcal{S}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{T}_x^\vee$ with respect to the differential $d_3 + d_4$ is $\mathcal{O}_{X,x}$ in degree 0, and 0 in other degrees. But this complex is the tensor product of the Koszul complexes $\mathrm{Koszul}_3(\mathcal{V}_x)$ (with the internal grading opposite to that in (2.3.3)) and $\mathrm{Koszul}_2(\mathcal{W}_x^\vee)$ of (2.3.2), both living in non-negative degrees. We have seen that these complexes have cohomology $\mathcal{O}_{X,x}$, and their components are free (hence flat). The result follows, using Künneth formula. □

2.6 Second generalized Koszul complex

Consider now the object

$$\mathcal{K}^{(2)} := \mathscr{A}(\mathcal{S}) \in \mathcal{C}_{\mathrm{gr}}^{\nwarrow}(\mathcal{T}).$$

It is concentrated in non-positive cohomological degrees, and in non-negative internal degrees. As in 2.5, we are going to prove:

Lemma 2.6.1. *The natural morphism $\mathcal{K}^{(2)} \to \mathcal{O}_X$ (projection on the $(0,0)$-component) is a quasi-isomorphism of $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{T}$-dg-modules.*

Proof. The arguments for this proof are completely similar to those of Lemma 2.5.1. Here the double complex to consider has (p,q)-term

$$C^{p,q} := \bigoplus_{\substack{p=-i-l, \\ q=-k-l}} \Lambda^i(\mathcal{V}_x) \otimes_{\mathcal{O}_{X,x}} \mathrm{S}^j(\mathcal{W}_x) \otimes_{\mathcal{O}_{X,x}} (\Lambda^k(\mathcal{W}_x^\vee))^\vee \otimes_{\mathcal{O}_{X,x}} (\mathrm{S}^l(\mathcal{V}_x^\vee))^\vee$$

and differentials $d' = d_1 + d_2$, $d'' = d_3 + d_4$. We have $C^{p,q} = 0$ for $p > 0$. □

3 Algebraic duality

In this section we prove our Koszul duality between $\mathcal{S}$- and $\mathcal{T}$-dg-modules.

3.1 Resolutions

First we need to prove the existence of some resolutions.

Proposition 3.1.1. (i) *Let $\mathcal{M}$ be an object of $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S})$. There exist an object $\mathcal{P}$ of $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S})$ such that, for all indices i and j, $\mathcal{P}^i_j$ is $\mathcal{O}_X$-locally free of finite rank, and a quasi-isomorphism of $\mathcal{S}$-dg-modules $\mathcal{P} \xrightarrow{\mathrm{qis}} \mathcal{M}$.*

(ii) *Let $\mathcal{N}$ be an object of $\mathcal{C}^{\nwarrow}_{\mathrm{gr}}(\mathcal{T})$. There exist an object $\mathcal{Q}$ of $\mathcal{C}^{\nwarrow}_{\mathrm{gr}}(\mathcal{T})$ such that, for all indices i and j, $\mathcal{Q}^i_j$ is $\mathcal{O}_X$-locally free of finite rank, and a quasi-isomorphism of $\mathcal{T}$-dg-modules $\mathcal{Q} \xrightarrow{\mathrm{qis}} \mathcal{N}$.*

Proof. We give a proof only for point (i). The proof of (ii) is similar[4]. Let $\mathcal{M}$ be an object of $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S})$. Let N_1 and N_2 be integers such that $\mathcal{M}^i_j = 0$ for $i < N_1$ or $i + j > N_2$. First we consider $\mathcal{M}$ as a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module. Then, for each $j \leq N_2 - N_1$, $\mathcal{M}_j$ is a complex of coherent $\mathcal{O}_X$-modules, with non-zero terms only in the interval $[N_1, N_2 - j]$ (and $\mathcal{M}_j = 0$ otherwise). Using a standard procedure (see *e.g.* [Har66, I.4.6] and [Har77, III.Ex.6.9]), there exists a complex $\mathcal{L}_j$ of locally free $\mathcal{O}_X$-modules of finite rank, with non-zero terms only in the interval $[N_1, N_2 - j]$, and a surjective morphism of $\mathcal{O}_X$-dg-modules $\mathcal{L}_j \twoheadrightarrow \mathcal{M}_j$. Then $\mathcal{L} := \bigoplus_j \mathcal{L}_j$ is an object of $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{O}_X)$, and there is a surjective morphism of $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-modules $\mathcal{L} \twoheadrightarrow \mathcal{M}$. Then $\mathcal{P}^{(1)} := \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{L}$, endowed with the natural differential and the natural action of $\mathcal{S}$, is an object of $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S})$, and there is a surjective morphism of $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{S}$-dg-modules

$$\mathcal{P}^{(1)} \twoheadrightarrow \mathcal{M}.$$

Taking the kernel of this morphism, and repeating the procedure, we obtain objects $\mathcal{P}^{(i)}$ $(i = 1, \cdots, d)$ of $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S})$, (recall that $d = \dim(X)$) whose homogeneous components are locally free of finite rank over $\mathcal{O}_X$, and an exact sequence of $\mathcal{S}$-dg-modules

$$\mathcal{P}^{(d)} \to \mathcal{P}^{(d-1)} \to \cdots \to \mathcal{P}^{(1)} \to \mathcal{M} \to 0.$$

We define $\mathcal{P}^{(d+1)} := \ker(\mathcal{P}^{(d)} \to \mathcal{P}^{(d-1)})$. Then, for any indices i, j, the exact sequence

$$0 \to (\mathcal{P}^{(d+1)})^i_j \to \cdots \to (\mathcal{P}^{(1)})^i_j \to \mathcal{M}^i_j \to 0$$

is a resolution of the $\mathcal{O}_X$-coherent sheaf $\mathcal{M}^i_j$, the terms $(\mathcal{P}^{(k)})^i_j$ being locally free of finite rank for $k = 1, \cdots, d$. It follows that $(\mathcal{P}^{(d+1)})^i_j$ is also locally free of finite rank over $\mathcal{O}_X$ (see again [Har77, III.Ex.6.9]).

[4] One could also use the "regrading trick" of 3.5 below to show that these two statements are equivalent.

Finally we take

$$\mathcal{P} := \mathrm{Tot}\big(0 \to \mathcal{P}^{(d+1)} \to \mathcal{P}^{(d)} \to \cdots \to \mathcal{P}^{(1)} \to 0\big).$$

It is naturally an object of $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S})$, and an easy spectral sequence argument shows that the natural morphism $\mathcal{P} \to \mathcal{M}$ is a quasi-isomorphism of $\mathcal{S}$-dg-modules. □

3.2 Derived functors

Let us introduce some notation. If $\mathcal{A}$ is any quasi-coherent $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebra, we denote by $\mathcal{H}^{*}_{\mathrm{gr}}(\mathcal{A})$ the homotopy category of the category $\mathcal{C}^{*}_{\mathrm{gr}}(\mathcal{A})$, where $* = \nearrow, \nwarrow, \swarrow, \searrow$. The objects of $\mathcal{H}^{*}_{\mathrm{gr}}(\mathcal{A})$ are the same as those of $\mathcal{C}^{*}_{\mathrm{gr}}(\mathcal{A})$, and the morphisms in $\mathcal{H}^{*}_{\mathrm{gr}}(\mathcal{A})$ are the quotient of the morphisms in $\mathcal{C}^{*}_{\mathrm{gr}}(\mathcal{A})$ by the homotopy relation. These categories are naturally triangulated. We denote by $\mathcal{D}^{*}_{\mathrm{gr}}(\mathcal{A})$, the localization of $\mathcal{H}^{*}_{\mathrm{gr}}(\mathcal{A})$ with respect to quasi-isomorphisms.

As a corollary of Proposition 3.1.1, we obtain the following result.

Corollary 3.2.1. *The functors $\mathscr{A}$ and $\mathscr{B}$ admit derived functors (in the sense of Deligne)*

$$\overline{\mathscr{A}} : \mathcal{D}^{\searrow}_{\mathrm{gr}}(\mathcal{S}) \to \mathcal{D}^{\nwarrow}_{\mathrm{gr}}(\mathcal{T}), \qquad \overline{\mathscr{B}} : \mathcal{D}^{\nwarrow}_{\mathrm{gr}}(\mathcal{T}) \to \mathcal{D}^{\searrow}_{\mathrm{gr}}(\mathcal{S}).$$

Remark 3.2.2. The functor $\overline{\mathscr{A}}$ is the left derived functor of $\mathscr{A}$ if we consider it as a covariant functor $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S}) \to \mathcal{C}^{\nwarrow}_{\mathrm{gr}}(\mathcal{T})^{\mathrm{opp}}$, or the right derived functor of $\mathscr{A}$ if we consider it as a covariant functor $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S})^{\mathrm{opp}} \to \mathcal{C}^{\nwarrow}_{\mathrm{gr}}(\mathcal{T})$.

Proof. Case of the functor $\mathscr{A}$. To fix notations, in this proof we consider $\mathscr{A}$ as a covariant functor $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S}) \to \mathcal{C}^{\nwarrow}_{\mathrm{gr}}(\mathcal{T})^{\mathrm{opp}}$. To prove that $\mathscr{A}$ admits a left derived functor, it is enough to prove that there are enough objects split on the left[5] for $\mathscr{A}$ in the category $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S})$ (see [Del73] or [Kel96]). To prove the latter fact, using Proposition 3.1.1(i), it is enough to prove that if $f : \mathcal{P} \to \mathcal{Q}$ is a quasi-isomorphism between two objects of $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S})$ whose homogeneous components are $\mathcal{O}_X$-locally free of finite rank, then the induced morphism

$$\mathscr{A}(f) : \mathscr{A}(\mathcal{P}) \to \mathscr{A}(\mathcal{Q})$$

is again a quasi-isomorphism. Taking cones, this amounts to proving that if $\mathcal{P}$ is an acyclic object of $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S})$ whose homogeneous components are $\mathcal{O}_X$-locally free of finite rank, then $\mathscr{A}(\mathcal{P})$ is again acyclic.

So, let $\mathcal{P}$ be such a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{S}$-dg-module. For each index j, the complex of $\mathcal{O}_X$-modules $\mathcal{P}_j$ is acyclic, bounded, and all its components are locally free of finite rank. It follows that $\mathcal{P}^{\vee}$ is also acyclic. Let x be a point of X, and let us prove that $\mathscr{A}(\mathcal{P})_x$ is

[5] Recall (see *e.g.* III.1.4) that an object $\mathcal{M}$ of $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S})$ is *split on the left* for $\mathscr{A}$ if for any quasi-isomorphism $\mathcal{M}' \xrightarrow{\mathrm{qis}} \mathcal{M}$, there exists an object $\mathcal{M}''$ of $\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S})$ and a quasi-isomorphism $\mathcal{M}'' \xrightarrow{\mathrm{qis}} \mathcal{M}'$ such that the induced morphism $\mathscr{A}(\mathcal{M}'') \to \mathscr{A}(\mathcal{M})$ is again a quasi-isomorphism.

acyclic. We use the same notations as in 2.4. In particular, $d_{\mathscr{A}(\mathcal{P})}$ is the sum of four terms d_1, d_2, d_3 and d_4. We have an isomorphism

$$\mathscr{A}(\mathcal{P})_x \cong \bigoplus_{i,j,k,l} \Lambda^i(\mathcal{V}_x) \otimes_{\mathcal{O}_{X,x}} \mathrm{S}^j(\mathcal{W}_x) \otimes_{\mathcal{O}_{X,x}} (\mathcal{P}_x^\vee)_l^k,$$

where the term $\Lambda^i(\mathcal{V}_x) \otimes_{\mathcal{O}_{X,x}} \mathrm{S}^j(\mathcal{W}_x) \otimes_{\mathcal{O}_{X,x}} (\mathcal{P}_x^\vee)_l^k$ is in cohomological degree $k-i$. The effect of the differentials on the indices i,j,k,l may be described as

$$d_1 : \left\{ \begin{array}{ccc} i & \mapsto & i-1 \\ j & \mapsto & j+1 \end{array} \right. , \quad d_2 : k \mapsto k+1, \quad d_3 : \left\{ \begin{array}{ccc} i & \mapsto & i+1 \\ k & \mapsto & k+2 \\ l & \mapsto & l-2 \end{array} \right. , \quad d_4 : \left\{ \begin{array}{ccc} j & \mapsto & j+1 \\ k & \mapsto & k+1 \\ l & \mapsto & l-2 \end{array} \right. .$$

Hence, disregarding the internal grading, $\mathscr{A}(\mathcal{P})_x$ is the total complex of the double complex with (p,q)-term

$$C^{p,q} := \bigoplus_{\substack{p=-i-j-l, \\ q=k+l+j}} \Lambda^i(\mathcal{V}_x) \otimes_{\mathcal{O}_{X,x}} \mathrm{S}^j(\mathcal{W}_x) \otimes_{\mathcal{O}_{X,x}} (\mathcal{P}_x^\vee)_l^k,$$

with differentials $d' = d_3 + d_4$ and $d'' = d_1 + d_2$. By definition, $\mathcal{P}$ is in $\mathcal{C}_{\mathrm{gr}}^{\searrow}(\mathcal{S})$, hence $(\mathcal{P}^\vee)_l^k = 0$ for $k + l \ll 0$. Hence $C^{p,q} = 0$ for $q \ll 0$. By Proposition 2.2.1, it follows that there is a converging spectral sequence

$$E_1^{p,q} = H^q(C^{p,*}, d'') \Rightarrow H^{p+q}(\mathscr{A}(\mathcal{P})_x).$$

Hence we can forget about the differentials d_3 and d_4, *i.e.* it is sufficient to prove that the tensor product of $\mathcal{O}_{X,x}$-dg-modules

$$\mathcal{T}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{P}_x^\vee$$

is acyclic. We have seen above that $\mathcal{P}_x^\vee$ is acyclic, and $\mathcal{T}_x$ is a bounded complex of flat $\mathcal{O}_{X,x}$-modules. Hence $\mathcal{T}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{P}_x^\vee$ is indeed acyclic, which finishes the proof of the existence of the derived functor

$$\overline{\mathscr{A}} : \mathcal{D}_{\mathrm{gr}}^{\searrow}(\mathcal{S}) \to \mathcal{D}_{\mathrm{gr}}^{\nwarrow}(\mathcal{T}).$$

Case of the functor $\mathscr{B}$. The proof for the functor $\mathscr{B}$ is very similar. If $\mathcal{Q}$ is a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{T}$-dg-module as in Proposition 3.1.1(ii) which is acyclic, and $x \in X$, then we have

$$\mathscr{B}(\mathcal{Q})_x = \bigoplus_{i,j,k,l} \Lambda^i(\mathcal{W}_x^\vee) \otimes_{\mathcal{O}_{X,x}} \mathrm{S}^j(\mathcal{V}_x^\vee) \otimes_{\mathcal{O}_{X,x}} (\mathcal{Q}_x^\vee)_l^k,$$

where the term $\Lambda^i(\mathcal{W}_x^\vee) \otimes_{\mathcal{O}_{X,x}} \mathrm{S}^j(\mathcal{V}_x^\vee) \otimes_{\mathcal{O}_{X,x}} (\mathcal{Q}_x^\vee)_l^k$ is in cohomological degree $i + 2j + k$. Again $\mathcal{Q}^\vee$ is acyclic, and $d_{\mathscr{B}(\mathcal{N})}$ is the sum of four terms d_1, d_2, d_3 and d_4, whose effect on the indices i,j,k,l may be described as

$$d_1 : \left\{ \begin{array}{ccc} i & \mapsto & i-1 \\ j & \mapsto & j+1 \end{array} \right. , \quad d_2 : k \mapsto k+1, \quad d_3 : \left\{ \begin{array}{ccc} j & \mapsto & j+1 \\ k & \mapsto & k-1 \\ l & \mapsto & l+2 \end{array} \right. , \quad d_4 : \left\{ \begin{array}{ccc} i & \mapsto & i+1 \\ l & \mapsto & l+2 \end{array} \right. .$$

Hence, disregarding the internal grading, $\mathscr{B}(\mathcal{Q})_x$ is the total complex of the double complex with (p,q)-term

$$D^{p,q} := \bigoplus_{\substack{p=i+j,\\ q=k+j}} \Lambda^i(\mathcal{W}_x^\vee) \otimes_{\mathcal{O}_{X,x}} \mathrm{S}^j(\mathcal{V}_x^\vee) \otimes_{\mathcal{O}_{X,x}} (\mathcal{Q}_x^\vee)_l^k,$$

and with differentials $d' = d_3 + d_4$, $d'' = d_1 + d_2$. We know that $(\mathcal{Q}_x^\vee)_l^k = 0$ if $k \ll 0$, hence $D^{p,q} = 0$ for $p \ll 0$. By Proposition 2.2.1, it follows that there is a converging spectral sequence

$$E_1^{p,q} = H^q(D^{p,*}, d'') \Rightarrow H^{p+q}(\mathscr{B}(\mathcal{Q})_x).$$

Hence it is sufficient to prove that the tensor product of $\mathcal{O}_{X,x}$-dg-modules

$$\mathcal{S}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{Q}_x^\vee$$

is acyclic.

The ($\mathbb{G}_{\mathbf{m}}$-equivariant) $\mathcal{O}_{X,x}$-dg-module $\mathcal{S}_x$ has a finite filtration with subquotients finite numbers of copies of $\mathrm{S}(\mathcal{V}_x^\vee)$. Hence it is enough to prove that $\mathrm{S}(\mathcal{V}_x^\vee) \otimes_{\mathcal{O}_{X,x}} \mathcal{Q}_x^\vee$ is acyclic. But $\mathrm{S}(\mathcal{V}_x^\vee)$, as a ($\mathbb{G}_{\mathbf{m}}$-equivariant) $\mathcal{O}_{X,x}$-dg-module, is a direct sum of flat $\mathcal{O}_{X,x}$-modules (placed in different degrees), hence the latter fact is clear. □

3.3 Morphisms of functors

In this subsection we construct some morphisms of functors. We will prove in the next subsection that they are isomorphisms, which implies that $\overline{\mathscr{A}}$ and $\overline{\mathscr{B}}$ are equivalences of categories.

Proposition 3.3.1. *There exist natural morphisms of functors*

$$\overline{\mathscr{B}} \circ \overline{\mathscr{A}} \to \mathrm{Id}_{\mathcal{D}_{\mathrm{gr}}^{\searrow}(\mathcal{S})}, \qquad \overline{\mathscr{A}} \circ \overline{\mathscr{B}} \to \mathrm{Id}_{\mathcal{D}_{\mathrm{gr}}^{\nwarrow}(\mathcal{T})}.$$

Proof. Let us give the details for the first morphism. The construction of the second one is similar. It is sufficient to construct this morphism for any $\mathcal{A}$-dg-module $\mathcal{P}$ as in Proposition 3.1.1(i). In this case $\overline{\mathscr{A}}(\mathcal{P})$ is isomorphic to the image of $\mathscr{A}(\mathcal{P})$ in the derived category. As $\mathscr{A}(\mathcal{P})$ has also $\mathcal{O}_X$-locally free homogeneous components, $\overline{\mathscr{B}} \circ \overline{\mathscr{A}}(\mathcal{P})$ is isomorphic to the image of $\mathscr{B} \circ \mathscr{A}(\mathcal{P})$ in the derived category. We will define a morphism in $\mathcal{C}_{\mathrm{gr}}^{\searrow}(\mathcal{S})$

$$\mathscr{B} \circ \mathscr{A}(\mathcal{P}) \to \mathcal{P}. \tag{3.3.2}$$

First we begin with the following lemma, which can be checked by direct computation, using the isomorphisms (2.1.1) and (2.1.2).

Lemma 3.3.3. *As a bigraded $\mathcal{O}_X$-module, $(\mathscr{A}(\mathcal{P}))^\vee$ is naturally isomorphic to $\mathcal{T}^\vee \otimes_{\mathcal{O}_X} \mathcal{P}$. Under this isomorphism, locally around a point $x \in X$, with the notation of* (2.4.1)*, the differential becomes*

$$d_{(\mathscr{A}(\mathcal{P}))^\vee}(f \otimes p) = d(f) \otimes p + (-1)^{|f|} f \otimes d(p) \\ - (-1)^{|f|}\big(\sum_\alpha f \cdot v_\alpha \otimes v_\alpha^* \cdot p + \sum_\beta f \cdot w_\beta \otimes w_\beta^* \cdot p\big),$$

where we set $(f \cdot t)(t') = f(t \cdot t')$ *for* $f \in \mathcal{T}^\vee$ *and* $t, t' \in \mathcal{T}$.

Under the isomorphism of Lemma 3.3.3, we have as bigraded $\mathcal{O}_X$-modules

$$\mathscr{B} \circ \mathscr{A}(\mathcal{P}) \cong \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{T}^\vee \otimes_{\mathcal{O}_X} \mathcal{P}.$$

We define the morphism of bigraded $\mathcal{O}_X$-modules

$$\left\{ \begin{array}{ccc} \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{T}^\vee \otimes_{\mathcal{O}_X} \mathcal{P} & \to & \mathcal{P} \\ s \otimes f \otimes p & \mapsto & f(1_{\mathcal{T}}) \cdot s \cdot p \end{array} \right. .$$

This morphism clearly commutes with the $\mathcal{S}$-actions. Moreover, using Lemma 3.3.3, one easily checks that it also commutes with the differentials, hence defines the desired morphism (3.3.2). □

3.4 Equivalences

Theorem 3.4.1. *The functors* $\overline{\mathscr{A}}$, $\overline{\mathscr{B}}$ *are equivalences of categories, quasi-inverse to each other.*

Proof. First step: isomorphism $\overline{\mathscr{B}} \circ \overline{\mathscr{A}} \xrightarrow{\sim} \mathrm{Id}$. In Proposition 3.3.1, we have constructed a morphism of functors $\overline{\mathscr{B}} \circ \overline{\mathscr{A}} \to \mathrm{Id}$. In this first step we prove that it is an isomorphism. Let $\mathcal{P}$ be an object of $\mathcal{C}_{\mathrm{gr}}^{\searrow}(\mathcal{S})$ as in Proposition 3.1.1(i). We have seen in 3.3 that $\overline{\mathscr{B}} \circ \overline{\mathscr{A}}(\mathcal{P})$ is isomorphic to the image of $\mathscr{B} \circ \mathscr{A}(\mathcal{P})$ in the derived category. By Proposition 3.1.1(i), it is thus enough to prove that the induced morphism

$$\phi : \mathscr{B} \circ \mathscr{A}(\mathcal{P}) \to \mathcal{P}$$

is a quasi-isomorphism. Let us construct a section (over $\mathcal{O}_X$) for this morphism. As a bigraded $\mathcal{O}_X$-module we have $\mathscr{B} \circ \mathscr{A}(\mathcal{P}) \cong \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{T}^\vee \otimes_{\mathcal{O}_X} \mathcal{P}$. Let $\epsilon_{\mathcal{T}} \in \mathcal{T}^\vee$ be the unit section in $(\mathcal{T}^\vee)^0_0 = \mathcal{O}_X$. Now consider the morphism

$$\psi : \left\{ \begin{array}{ccc} \mathcal{P} & \to & \mathscr{B} \circ \mathscr{A}(\mathcal{P}) \\ p & \mapsto & 1_{\mathcal{S}} \otimes \epsilon_{\mathcal{T}} \otimes p \end{array} \right. .$$

One easily checks that it is a morphism of $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-modules (but of course not of $\mathcal{S}$-dg-modules), and that

$$\phi \circ \psi = \mathrm{Id}_{\mathcal{P}}.$$

Hence it is enough to prove that ψ is a quasi-isomorphism.

As a bigraded $\mathcal{O}_X$-module, we have, with the notation of 2.5,

$$\mathscr{B} \circ \mathscr{A}(\mathcal{P}) \cong \mathcal{K}^{(1)} \otimes_{\mathcal{O}_X} \mathcal{P} \cong \bigoplus_{i,j,k,l} (\mathcal{K}^{(1)})^i_k \otimes_{\mathcal{O}_X} \mathcal{P}^j_l,$$

where the term $(\mathcal{K}^{(1)})^i_k \otimes_{\mathcal{O}_X} \mathcal{P}^j_l$ is in cohomological degree $i+j$. Remark that here the nonzero terms occur only when k is even. By Lemma 3.3.3, the differential on $\mathscr{B} \circ \mathscr{A}(\mathcal{P})$ is the sum of four terms. The first one is $d_1 := d_{\mathcal{K}^{(1)}} \otimes \mathrm{Id}_{\mathcal{P}}$. The second one is $d_2 := \mathrm{Id}_{\mathcal{K}^{(1)}} \otimes d_{\mathcal{P}}$. The third one is the "Koszul type" differential coming from the left action of $\mathcal{V}^\vee \subset \mathcal{S}$ on $\mathcal{P}$ and the right action of $\mathcal{V} \subset \mathcal{T}$ on $\mathcal{K}^{(1)}$. Finally d_4 is the similar "Koszul-type" differential coming from the actions of $\mathcal{W}^\vee$ and $\mathcal{W}$. The effect of these differentials on the indices i, j, k, l can be described as follows:

$$d_1 : i \to i+1, \quad d_2 : j \mapsto j+1, \quad d_3 : \begin{cases} i & \mapsto & i-1 \\ j & \mapsto & j+2 \\ k & \mapsto & k+2 \\ l & \mapsto & l-2 \end{cases}, \quad d_4 : \begin{cases} j & \mapsto & j+1 \\ k & \mapsto & k+2 \\ l & \mapsto & l-2 \end{cases}.$$

Moreover, one easily checks the following relations:

$$(d_1 + d_4)^2 = 0, \qquad (d_2 + d_3)^2 = 0.$$

Hence, disregarding the internal grading, $\mathscr{B} \circ \mathscr{A}(\mathcal{P})$ is the total complex of the double complex with (p,q)-term

$$C^{p,q} := \bigoplus_{\substack{p=j+l+k/2, \\ q=i-l-k/2}} (\mathcal{K}^{(1)})^i_k \otimes_{\mathcal{O}_X} \mathcal{P}^j_l,$$

and with differentials $d' = d_2 + d_3$ and $d'' = d_1 + d_4$. We know that $\mathcal{P}^j_l = 0$ for $j + l \gg 0$, and that $(\mathcal{K}^{(1)})^i_k = 0$ if $k > 0$. Hence $C^{p,q} = 0$ for $p \gg 0$. It follows, by Proposition 2.2.1, that there is a converging spectral sequence

$$E_1^{p,q} = H^q(C^{p,*}, d'') \Rightarrow H^{p+q}(\mathscr{B} \circ \mathscr{A}(\mathcal{P})).$$

Disregarding the internal grading, $\mathcal{P}$ is also the total complex of a double complex, defined by

$$(C')^{p,q} := \mathcal{P}^{p+q}_{-q}$$

and the differentials $d' = d_{\mathcal{P}}$, $d'' = 0$. Here also $(C')^{p,q} = 0$ for $p \gg 0$, hence the corresponding spectral sequence converges. Moreover, ψ is induced by a morphism of double complexes $C' \to C$. It follows that it is enough to prove that the morphism induced by ψ from $\mathcal{P}$, endowed with the zero differential, to $\mathcal{K}^{(1)} \otimes_{\mathcal{O}_X} \mathcal{P}$, endowed with the differential $d_1 + d_4$, is a quasi-isomorphism.

The latter dg-module is again the total complex of the double complex with (p,q)-term

$$D^{p,q} := \bigoplus_{k,l} (\mathcal{K}^{(1)})^q_k \otimes_{\mathcal{O}_X} \mathcal{P}^p_l,$$

and differentials $d' = d_4$, $d'' = d_1$. And $\mathcal{P}$ (with the trivial differential) is also the total complex of the double complex defined by

$$(D')^{p,q} = \begin{cases} \bigoplus_l \mathcal{P}^p_l & \text{if } q = 0, \\ 0 & \text{otherwise}, \end{cases}$$

and with two trivial differentials. Again ψ is induced by a morphism of double complexes, and we have $D^{p,q} = (D')^{p,q} = 0$ for $q < 0$. We conclude that the associated spectral sequences converge. As $H(\mathcal{K}_1) = \mathcal{O}_X$ (see Lemma 2.5.1) and $\mathcal{P}$ is a bounded above complex of flat $\mathcal{O}_X$-modules, we finally conclude that ψ is a quasi-isomorphism.

Second step: isomorphism $\overline{\mathscr{A}} \circ \overline{\mathscr{B}} \xrightarrow{\sim} \mathrm{Id}$. The proofs in this second step are very similar to those of the first step. By Proposition 3.3.1 there is a natural morphism $\overline{\mathscr{A}} \circ \overline{\mathscr{B}} \to \mathrm{Id}$, and we prove that it is an isomorphism. As above, it is enough to prove that, for $\mathcal{Q}$ an object of $\mathcal{C}_{\mathrm{gr}}^{\nwarrow}(\mathcal{T})$ as in Proposition 3.1.1(ii), the induced morphism of $\mathcal{T}$-dg-modules

$$\phi : \mathscr{A} \circ \mathscr{B}(\mathcal{Q}) \to \mathcal{Q}$$

is a quasi-isomorphism. Also as above one can construct a section

$$\psi : \mathcal{Q} \to \mathscr{A} \circ \mathscr{B}(\mathcal{Q})$$

of ϕ as a morphism of $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-modules, and it is enough to prove that ψ is a quasi-isomorphism.

Here we have as bigraded $\mathcal{O}_X$-modules, with the notation of 2.6,

$$\mathscr{A} \circ \mathscr{B}(\mathcal{Q}) \cong \mathcal{K}^{(2)} \otimes_{\mathcal{O}_X} \mathcal{Q} \cong \bigoplus_{i,j,k,l} (\mathcal{K}^{(2)})^i_k \otimes_{\mathcal{O}_X} \mathcal{Q}^j_l,$$

where $(\mathcal{K}^{(2)})^i_k \otimes_{\mathcal{O}_X} \mathcal{Q}^j_l$ is in cohomological degree $i+j$ (and k is even if the term is non-zero). Again the differential is the sum of four terms $d_1 := d_{\mathcal{K}^{(2)}} \otimes \mathrm{Id}_{\mathcal{Q}}$, $d_2 = \mathrm{Id}_{\mathcal{K}^{(2)}} \otimes d_{\mathcal{Q}}$, d_3 the Koszul differential induced by the action of $\mathcal{V}$ and $\mathcal{V}^\vee$, and d_4 the Koszul differential induced by the action of $\mathcal{W}$ and $\mathcal{W}^\vee$. The effect of these differentials on the indices i, j, k, l can be described as follows:

$$d_1 : i \to i+1, \quad d_2 : j \mapsto j+1, \quad d_3 : \begin{cases} i & \mapsto & i+2 \\ j & \mapsto & j-1 \\ k & \mapsto & k-2 \\ l & \mapsto & l+2 \end{cases}, \quad d_4 : \begin{cases} i & \mapsto & i+1 \\ k & \mapsto & k-2 \\ l & \mapsto & l+2 \end{cases}.$$

One has

$$(d_1 + d_2)^2 = 0, \qquad (d_3 + d_4)^2 = 0.$$

Hence, disregarding the internal grading, $\mathscr{A} \circ \mathscr{B}(\mathcal{Q})$ is the total complex of the double complex with (p, q)-term

$$C^{p,q} := \bigoplus_{\substack{p=-l-3k/2, \\ q=i+j+l+3k/2}} (\mathcal{K}^{(2)})^i_k \otimes_{\mathcal{O}_X} \mathcal{Q}^j_l,$$

and with differentials $d' = d_3 + d_4$, $d'' = d_1 + d_2$. We know that $\mathcal{Q}^j_l = 0$ if $j + l \ll 0$. Moreover, one checks easily that $(\mathcal{K}^{(2)})^i_k = 0$ if $i + 3k/2 \ll 0$. Hence $C^{p,q} = 0$ if $q \ll 0$. It follows, by Proposition 2.2.1, that there is a converging spectral sequence

$$E_1^{p,q} = H^q(C^{p,*}, d'') \Rightarrow H^{p+q}(\mathscr{A} \circ \mathscr{B}(\mathcal{Q})).$$

Similarly, disregarding the internal grading, $\mathcal{Q}$ is the total complex of a double complex C', and ψ is induced by a morphism of double complexes $C' \to C$. Hence it is enough to prove that the morphism induced by ψ from $\mathcal{Q}$ to $\mathcal{K}^{(2)} \otimes_{\mathcal{O}_X} \mathcal{Q}$, endowed with the differential $d_1 + d_2$, is a quasi-isomorphism.

Once more, this follows from a spectral sequence argument, using the property that $H(\mathcal{K}^{(2)}) = \mathcal{O}_X$ (see Lemma 2.6.1). □

3.5 Regrading

In this subsection we introduce a "regrading" functor. This functor will play a technical role in 3.6, and a more crucial role later in the geometric interpretation of the equivalence.

Consider the functor

$$\xi : \mathcal{C}_{\mathrm{gr}}(\mathcal{S}) \to \mathcal{C}_{\mathrm{gr}}(\mathcal{R})$$

which sends the $\mathcal{S}$-dg-module $\mathcal{M}$ to the $\mathcal{R}$-dg-module with (i,j)-component $\xi(\mathcal{M})^i_j := \mathcal{M}^{i-j}_j$, the differential and the $\mathcal{R}$-action on $\xi(\mathcal{M})$ being induced by the differential and the $\mathcal{S}$-action on $\mathcal{M}$. This functor is clearly an equivalence of categories, and it induces equivalences, still denoted ξ,

$$\mathcal{C}^{\searrow}_{\mathrm{gr}}(\mathcal{S}) \xrightarrow{\sim} \mathcal{C}^{\swarrow}_{\mathrm{gr}}(\mathcal{R}), \qquad \mathcal{D}^{\searrow}_{\mathrm{gr}}(\mathcal{S}) \xrightarrow{\sim} \mathcal{D}^{\swarrow}_{\mathrm{gr}}(\mathcal{R}).$$

3.6 Categories with finiteness conditions

In the rest of this section we prove that the equivalences $\overline{\mathscr{A}}$ and $\overline{\mathscr{B}}$ restrict to equivalences between subcategories of dg-modules whose cohomology is locally finitely generated. This will eventually allow us to get rid of the technical conditions "$\nwarrow$" and "$\searrow$".

Let us introduce some more notation. If $\mathcal{A}$ is a quasi-coherent $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebra, and if $* = \nwarrow, \swarrow, \searrow, \nearrow$, we denote by $\mathcal{C}^{*,\mathrm{fg}}_{\mathrm{gr}}(\mathcal{A})$, respectively $\mathcal{D}^{*,\mathrm{fg}}_{\mathrm{gr}}(\mathcal{A})$, the full subcategory of $\mathcal{C}^{*}_{\mathrm{gr}}(\mathcal{A})$, respectively $\mathcal{D}^{*}_{\mathrm{gr}}(\mathcal{A})$, whose objects are the dg-modules $\mathcal{M}$ such that $H(\mathcal{M})$ is a locally finitely generated $H(\mathcal{A})$-module.

We also denote by $\mathcal{CFG}_{\mathrm{gr}}(\mathcal{A})$ the full subcategory of $\mathcal{C}_{\mathrm{gr}}(\mathcal{A})$ whose objects are the locally finitely generated $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{A}$-dg-modules, and by $\mathcal{DFG}_{\mathrm{gr}}(\mathcal{A})$ the localization of the homotopy category of $\mathcal{CFG}_{\mathrm{gr}}(\mathcal{A})$ with respect to quasi-isomorphisms. Finally we denote by $\mathcal{D}^{\mathrm{fg}}_{\mathrm{gr}}(\mathcal{A})$ the full subcategory of $\mathcal{D}_{\mathrm{gr}}(\mathcal{A})$ whose objects are the $\mathbb{G}_{\mathbf{m}}$-equivariant dg-modules $\mathcal{M}$ such that $H(\mathcal{M})$ is locally finitely generated over $H(\mathcal{A})$.

We are going to prove that, in the cases we are interested in, several of these categories coincide. Observe in particular that there are inclusions

$$\mathcal{CFG}_{\mathrm{gr}}(\mathcal{R}) \hookrightarrow \mathcal{C}^{\swarrow,\mathrm{fg}}_{\mathrm{gr}}(\mathcal{R}), \quad \mathcal{CFG}_{\mathrm{gr}}(\mathcal{S}) \hookrightarrow \mathcal{C}^{\searrow,\mathrm{fg}}_{\mathrm{gr}}(\mathcal{S}), \quad \mathcal{CFG}_{\mathrm{gr}}(\mathcal{T}) \hookrightarrow \mathcal{C}^{\nwarrow,\mathrm{fg}}_{\mathrm{gr}}(\mathcal{T}),$$

which induce functors between the corresponding derived categories.

Lemma 3.6.1. (i) *The induced functors*

$$\mathcal{DFG}_{\mathrm{gr}}(\mathcal{R}) \to \mathcal{D}^{\swarrow,\mathrm{fg}}_{\mathrm{gr}}(\mathcal{R}), \quad \mathcal{DFG}_{\mathrm{gr}}(\mathcal{S}) \to \mathcal{D}^{\searrow,\mathrm{fg}}_{\mathrm{gr}}(\mathcal{S}), \quad \mathcal{DFG}_{\mathrm{gr}}(\mathcal{T}) \to \mathcal{D}^{\nwarrow,\mathrm{fg}}_{\mathrm{gr}}(\mathcal{T})$$

are equivalences of categories.

(ii) *Similarly, the natural functors*

$$\mathcal{DFG}_{\mathrm{gr}}(\mathcal{R}) \to \mathcal{D}^{\mathrm{fg}}_{\mathrm{gr}}(\mathcal{R}), \quad \mathcal{DFG}_{\mathrm{gr}}(\mathcal{S}) \to \mathcal{D}^{\mathrm{fg}}_{\mathrm{gr}}(\mathcal{S}), \quad \mathcal{DFG}_{\mathrm{gr}}(\mathcal{T}) \to \mathcal{D}^{\mathrm{fg}}_{\mathrm{gr}}(\mathcal{T})$$

are equivalences of categories.

Proof. Our proof of this lemma is very similar to that of [Bor87, VI.2.11] (see also Proposition III.3.2.4). We give the details of the proof of (ii). Statement (i) can be treated similarly.

Using the "regrading trick" of 3.5, the cases of $\mathcal{S}$ and $\mathcal{R}$ are equivalent. Similarly, using the change of the internal grading to the opposite one, we see that the cases of $\mathcal{R}$ and $\mathcal{T}$ are equivalent. Hence it is sufficient to consider the $\mathbb{G}_{\mathbf{m}}$-equivariant dg-algebra $\mathcal{T}$.

Remark that the algebra $\mathcal{T}$, as well as its cohomology $H(\mathcal{T})$, is finitely generated as a $\mathrm{S}(\mathcal{W})$-module. Hence a $\mathcal{T}$-dg-module $\mathcal{N}$ is locally finitely generated, respectively has locally finitely generated cohomology, iff $\mathcal{N}$, respectively $H(\mathcal{N})$, is locally finitely generated over $\mathrm{S}(\mathcal{W})$.

Lemma 3.6.2. *Let $\mathcal{N}$ be an object of $\mathcal{C}_{\mathrm{gr}}(\mathcal{T})$, with locally finitely generated cohomology, whose cohomological grading is bounded. Then $\mathcal{N}$ is the inductive limit of quasi-coherent sub-$\mathcal{T}$-dg-modules which are locally finitely generated, and which are quasi-isomorphic to $\mathcal{N}$ under inclusion.*

Proof of Lemma 3.6.2. The internal grading has no importance in this statement, hence we will forget about it in the proof. The dg-module $\mathcal{N}$ is clearly an inductive limit of locally finitely generated quasi-coherent sub-$\mathcal{T}$-dg-modules. Hence it is sufficient to show that given a locally finitely generated quasi-coherent sub-dg-module $\mathcal{F}$ of $\mathcal{N}$, there exists a locally finitely generated quasi-coherent sub-dg-module $\mathcal{G}$ of $\mathcal{N}$, containing $\mathcal{F}$ and quasi-isomorphic to $\mathcal{N}$ under the inclusion map.

This is proved by a simple (descending) induction. Let $j \in \mathbb{Z}$. Assume that we have found a subcomplex $\mathcal{G}_{(j)}$ of $\bigoplus_{i \geq j} \mathcal{N}^i$, quasi-coherent over $\mathcal{O}_X$, locally finitely generated over $\mathrm{S}(\mathcal{W})$, containing $\bigoplus_{i \geq j} \mathcal{F}^i$, stable under $\mathcal{T}$ (*i.e.* if $g \in \mathcal{G}^i_{(j)}$ and $t \in \mathcal{T}^k$, and if $i+k \geq j$, then $t \cdot g \in \mathcal{G}^{i+k}_{(j)}$), such that $\mathcal{G}_{(j)} \hookrightarrow \mathcal{N}$ is a quasi-isomorphism in cohomological degrees greater than j and that $\mathcal{G}^j_{(j)} \cap \ker(d^j_{\mathcal{N}}) \to H^j(\mathcal{N})$ is surjective. Then we choose a locally finitely generated sub-$\mathrm{S}(\mathcal{W})$-module $\mathcal{H}^{j-1}$ of $\mathcal{N}^{j-1}$ containing $\mathcal{F}^{j-1}$, quasi-coherent over $\mathcal{O}_X$, whose image under $d^{j-1}_{\mathcal{N}}$ is $\mathcal{G}^j_{(j)} \cap \mathrm{Im}(d^{j-1}_{\mathcal{N}})$. Without altering these conditions, we can add a sub-module of cocycles so that the new sub-module $\mathcal{H}^{j-1}$ contains representatives of all the elements of $H^{j-1}(\mathcal{N})$. We can also assume that $\mathcal{N}^{j-1}$ contains all the sections of the form $t \cdot g$ for $t \in \mathcal{T}^i$ and $g \in \mathcal{G}^k_{(j)}$ with $i + k = j - 1$. Then we define $\mathcal{G}_{(j-1)}$ by

$$\mathcal{G}^k_{(j-1)} = \begin{cases} \mathcal{G}^k_{(j)} & \text{if } k \geq j, \\ \mathcal{H}^{j-1} & \text{if } k = j-1. \end{cases}$$

For j small enough, $\mathcal{G}_{(j)}$ is the sought-for sub-dg-module. □

Let us denote by

$$\iota : \mathcal{DFG}_{\mathrm{gr}}(\mathcal{T}) \to \mathcal{D}^{\mathrm{fg}}_{\mathrm{gr}}(\mathcal{T})$$

the functor under consideration. Let $\mathcal{N}$ be an object of $\mathcal{D}^{\mathrm{fg}}_{\mathrm{gr}}(\mathcal{T})$. Then the cohomology $H(\mathcal{N})$ is bounded for the cohomological grading (because it is locally finitely generated over $H(\mathcal{T})$, which is bounded). Hence, using truncation functors (see 2.1), $\mathcal{N}$ is isomorphic to a $\mathcal{T}$-dg-module whose cohomological grading is bounded. Using Lemma 3.6.2, it follows that $\mathcal{N}$ is in the essential image of ι. Hence ι is essentially surjective.

Now, let us prove that it is full. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be objects of $\mathcal{CFG}_{\mathrm{gr}}(\mathcal{T})$. In particular, $\mathcal{N}_1$ and $\mathcal{N}_2$ have bounded cohomological grading. A morphism $\phi : \iota(\mathcal{N}_1) \to \iota(\mathcal{N}_2)$ in $\mathcal{D}^{\mathrm{fg}}_{\mathrm{gr}}(\mathcal{T})$ is represented by a diagram

$$\iota(\mathcal{N}_1) \xrightarrow{\alpha} \mathcal{F} \xleftarrow{\beta} \iota(\mathcal{N}_2)$$

where β is a quasi-isomorphism. Using truncation functors, one can assume that $\mathcal{F}$ has bounded cohomological grading. By Lemma 3.6.2, there exists a locally finitely generated sub-$\mathcal{T}$-dg-module $\mathcal{F}'$ of $\mathcal{F}$, containing $\alpha(\mathcal{N}_1)$ and $\beta(\mathcal{N}_2)$, and quasi-isomorphic to $\mathcal{F}$ under the inclusion map. Then ϕ is also represented by

$$\iota(\mathcal{N}_1) \xrightarrow{\alpha} \mathcal{F}' \xleftarrow{\beta} \iota(\mathcal{N}_2),$$

which is the image of a morphism in $\mathcal{DFG}_{\mathrm{gr}}(\mathcal{T})$. Hence ι is full.

Finally we prove that ι is faithful. If a morphism $f : \mathcal{N}_1 \to \mathcal{N}_2$ in $\mathcal{CFG}_{\mathrm{gr}}(\mathcal{T})$ is such that $\iota(f) = 0$, then there exist an object $\mathcal{F}$ of $\mathcal{D}^{\mathrm{fg}}_{\mathrm{gr}}(\mathcal{T})$, which can again be assumed to be bounded, and a quasi-isomorphism of $\mathcal{T}$-dg-modules $g : \mathcal{N}_2 \to \mathcal{F}$ such that $g \circ f$ is homotopic to zero. This homotopy is given by a morphism $h : \mathcal{N}_1 \to \mathcal{F}[-1]$. Again by Lemma 3.6.2, there exists a locally finitely generated sub-$\mathcal{T}$-dg-module $\mathcal{F}'$ of $\mathcal{F}$ containing $g(\mathcal{N}_2)$ and $h(\mathcal{N}_1)[1]$, and quasi-isomorphic to $\mathcal{F}$ under inclusion. Replacing $\mathcal{F}$ by $\mathcal{F}'$, this proves that $f = 0$ in $\mathcal{DFG}_{\mathrm{gr}}(\mathcal{T})$. The proof of Lemma 3.6.1 is complete. □

3.7 Restriction of the equivalences to locally finitely generated dg-modules

Proposition 3.7.1. *The functors $\overline{\mathscr{A}}$, $\overline{\mathscr{B}}$ restrict to equivalences of categories*

$$\mathcal{D}^{\searrow,\mathrm{fg}}_{\mathrm{gr}}(\mathcal{S}) \cong \mathcal{D}^{\nwarrow,\mathrm{fg}}_{\mathrm{gr}}(\mathcal{T}).$$

Proof. It is sufficient to prove that the functors $\overline{\mathscr{A}}$, $\overline{\mathscr{B}}$ send dg-modules with locally finitely generated cohomology to dg-modules with locally finitely generated cohomology.

First step: functor $\overline{\mathscr{B}}$. First, let us consider $\overline{\mathscr{B}}$. By Lemma 3.6.1, it suffices to prove that if $\mathcal{N}$ is a locally finitely generated $\mathcal{T}$-module, then $\overline{\mathscr{B}}(\mathcal{N})$ has locally finitely generated cohomology. We begin with the following lemma.

Lemma 3.7.2. *Let $\mathcal{N}$ be a locally finitely generated $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{T}$-dg-module. There exist an object $\mathcal{Q}$ of $\mathcal{CFG}_{\mathrm{gr}}(\mathcal{T})$, which is locally free of finite rank over $\mathrm{S}(\mathcal{W}) \subset \mathcal{T}$, and a quasi-isomorphism $\mathcal{Q} \xrightarrow{\mathrm{qis}} \mathcal{N}$.*

Proof of Lemma 3.7.2. The arguments in this proof are very close to those in the proof of Proposition 3.1.1. There exists a $\mathbb{G}_{\mathbf{m}}$-equivariant sub-$\mathcal{O}_X$-dg-module $\mathcal{G} \subset \mathcal{N}$, which is coherent as an $\mathcal{O}_X$-module, and which generates $\mathcal{N}$ as a $\mathcal{S}$-dg-module. There exists also a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module $\mathcal{F}$, which is locally free of finite rank as an $\mathcal{O}_X$-module, and a surjection $\mathcal{F} \twoheadrightarrow \mathcal{G}$. We set

$$\mathcal{Q}^{(1)} := \mathcal{T} \otimes_{\mathcal{O}_X} \mathcal{F},$$

endowed with its natural structure of $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{T}$-dg-module. Then we have a surjection of $\mathcal{T}$-dg-modules

$$\mathcal{Q}^{(1)} \twoheadrightarrow \mathcal{N},$$

and $\mathcal{Q}^{(1)}$ is locally free over $\mathrm{S}(\mathcal{W})$.

Let n be the rank of $\mathcal{W}$ over $\mathcal{O}_X$. Taking the kernel of our morphism $\mathcal{Q}^{(1)} \to \mathcal{N}$, and repeating the argument, we obtain locally finitely generated $\mathcal{T}$-dg-modules $\mathcal{Q}^{(j)}$, $j = 1, \cdots, n+d$, which are locally free of finite rank over $\mathrm{S}(\mathcal{W})$, and an exact sequence of $\mathcal{T}$-dg-modules

$$\mathcal{Q}^{(n+d)} \to \mathcal{Q}^{(d+n-1)} \to \cdots \to \mathcal{Q}^{(1)} \to \mathcal{N} \to 0.$$

All these objects are complexes of coherent $\mathcal{S}(\mathcal{W})$-modules, hence we can consider them as complexes of coherent sheaves on W^*, the vector bundle on X with sheaf of sections $\mathcal{W}^\vee$. The scheme W^* is noetherian, integral, separated and regular of dimension $d+n$. Hence $\mathcal{Q}^{(n+d+1)} := \mathrm{Ker}(\mathcal{Q}^{(n+d)} \to \mathcal{Q}^{(n+d-1)})$ is also locally free over $\mathrm{S}(\mathcal{W})$. Then

$$\mathcal{Q} := \mathrm{Tot}(0 \to \mathcal{Q}^{(n+d+1)} \to \cdots \to \mathcal{Q}^{(1)} \to 0)$$

is a resolution of $\mathcal{N}$ as in the lemma. □

Now let $\mathcal{Q} \xrightarrow{\mathrm{qis}} \mathcal{N}$ be a resolution as in Lemma 3.7.2. In particular $\mathcal{Q}$ is locally free over $\mathcal{O}_X$, hence $\overline{\mathscr{B}}(\mathcal{N})$ is isomorphic to the image of $\mathscr{B}(\mathcal{Q})$ in the derived category. Hence it is enough to prove that $\mathscr{B}(\mathcal{Q})$ has locally finitely generated cohomology, and even to prove that this cohomology is locally finitely generated over $\mathrm{S}(\mathcal{V}^\vee)$. Let $x \in X$. The object $\mathscr{B}(\mathcal{Q})_x$ was described in 3.2. We use the same notations as in this subsection. Disregarding the internal grading, $\mathscr{B}(\mathcal{Q})_x$ is also the total complex of the double complex with (p,q)-term

$$C^{p,q} := \bigoplus_{\substack{p=j,\\ q=i+k+j}} \Lambda^i(\mathcal{W}_x^\vee) \otimes_{\mathcal{O}_{X,x}} \mathrm{S}^j(\mathcal{V}_x^\vee) \otimes_{\mathcal{O}_{X,x}} (\mathcal{Q}_x^\vee)_l^k,$$

and with differentials $d' = d_1 + d_3$, $d'' = d_2 + d_4$. By hypothesis, $(\mathcal{Q}^\vee)_l^k = 0$ for $k \ll 0$, hence $C^{p,q} = 0$ for $q \ll 0$. Hence by Proposition 2.2.1 there is a converging spectral sequence

$$E_1^{p,q} = H(C^{p,*}, d'') \Rightarrow H^{p+q}(\mathscr{B}(\mathcal{Q})_x).$$

It follows that it is sufficient to prove that the cohomology of $\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{Q}^\vee$, endowed with the differential $d_2 + d_4$, is locally finitely generated over $\mathrm{S}(\mathcal{V}^\vee)$. This complex is again the total complex of the double complex with (p,q)-term

$$D^{p,q} := \bigoplus_{\substack{p=2j+k,\\ q=i}} \Lambda^i(\mathcal{W}_x^\vee) \otimes_{\mathcal{O}_{X,x}} \mathrm{S}^j(\mathcal{V}_x^\vee) \otimes_{\mathcal{O}_{X,x}} (\mathcal{Q}_x^\vee)_l^k,$$

and with differentials $d' = d_2$, $d'' = d_3$. The spectral sequence of this double complex again converges, hence we can forget about d_2. Then $\mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{Q}^\vee$, endowed with the differential d_3, is locally the tensor product of $\mathrm{S}(\mathcal{V}^\vee)$ with a finite number of Koszul complexes $\mathrm{Koszul}_2(\mathcal{W}_x^\vee)$ of (2.3.2). The result follows.

Second step: functor $\overline{\mathscr{A}}$. The proof for the functor $\overline{\mathscr{A}}$ is entirely similar. In this case, with the notation of 3.2, we can consider the double complex with (p,q)-term

$$C^{p,q} := \bigoplus_{\substack{p=k-2i-j,\\ q=i+j}} \Lambda^i(\mathcal{W}_x^\vee) \otimes_{\mathcal{O}_{X,x}} \mathrm{S}^j(\mathcal{V}_x^\vee) \otimes_{\mathcal{O}_{X,x}} (\mathcal{P}_x^\vee)_l^k,$$

and differentials $d' = d_1 + d_2$, $d'' = d_3 + d_4$. Here $C^{p,q} = 0$ for $q < 0$, hence the corresponding spectral sequence converges, and we can forget about d_1 and d_2. Then we can consider the double complex

$$D^{p,q} := \bigoplus_{\substack{p=k-2i,\\ q=i}} \Lambda^i(\mathcal{W}_x^\vee) \otimes_{\mathcal{O}_{X,x}} \mathrm{S}^j(\mathcal{V}_x^\vee) \otimes_{\mathcal{O}_{X,x}} (\mathcal{P}_x^\vee)_l^k,$$

with differentials $d' = d_4$ and $d'' = d_3$. And we finish the proof as above. □

Finally, combining Proposition 3.7.1, Lemma 3.6.1 and the "regrading trick" of 3.5 we obtain the following theorem, which is the main result of this section.

Theorem 3.7.3. *There exists a contravariant equivalence of triangulated categories*

$$\kappa : \mathcal{D}^{\mathrm{fg}}_{\mathrm{gr}}(\mathcal{T}) \xrightarrow{\sim} \mathcal{D}^{\mathrm{fg}}_{\mathrm{gr}}(\mathcal{R})$$

satisfying $\kappa(\mathcal{M}[n]\langle m\rangle) = \kappa(\mathcal{M})[-n+m]\langle m\rangle$.

4 Linear Koszul duality

In this section we give a geometric interpretation of Theorem 3.7.3.

4.1 Intersections of vector bundles

Let us consider as above a noetherian, integral, separated, regular scheme X, and a vector bundle E over X. Let $F_1, F_2 \subset E$ be sub-vector bundles. Let E^* be the vector bundle dual to E, and let $F_1^\perp, F_2^\perp \subset E^*$ be the orthogonal to F_1, respectively F_2. We will be interested in the dg-schemes

$$F_1 \overset{R}{\cap}_E F_2 \qquad \text{and} \qquad F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp.$$

Let $\mathcal{E}, \mathcal{F}_1, \mathcal{F}_2$ be the sheaves of local sections of E, F_1, F_2. Then the sheaves of local sections of $E^*, F_1^\perp, F_2^\perp$ are, respectively, $\mathcal{E}^\vee$, $\mathcal{F}_1^\perp$ and $\mathcal{F}_2^\perp$ (here we consider the orthogonals inside $\mathcal{E}^\vee$). Let us denote by $\mathcal{X}$ the $\mathcal{O}_X$-dg-module

$$\mathcal{X} := \big(0 \to \mathcal{F}_1^\perp \to \mathcal{F}_2^\vee \to 0\big),$$

where $\mathcal{F}_1^\perp$ is in degree -1, $\mathcal{F}_2^\vee$ is in degree 0, and the non-trivial differential is the composition of the natural morphisms $\mathcal{F}_1^\perp \hookrightarrow \mathcal{E}^\vee \twoheadrightarrow \mathcal{F}_2^\vee$, and by $\mathcal{Y}$ the $\mathcal{O}_X$-dg-module

$$\mathcal{Y} := \big(0 \to \mathcal{F}_2 \to \mathcal{E}/\mathcal{F}_1 \to 0\big),$$

where $\mathcal{F}_2$ is in degree -1, $\mathcal{E}/\mathcal{F}_1$ is in degree 0, and the non-trivial differential is the opposite of the composition of the natural morphisms $\mathcal{F}_2 \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{F}_1$.

Lemma 4.1.1. *There exist equivalences of categories*

$$\mathcal{D}(F_1 \overset{R}{\cap}_E F_2) \cong \mathcal{D}(X, \mathrm{Sym}(\mathcal{X})), \qquad \mathcal{D}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp) \cong \mathcal{D}(X, \mathrm{Sym}(\mathcal{Y})).$$

Proof. We need only prove the first equivalence (the second one is similar: replace E by E^*, F_1 by $F_2^\perp$, F_2 by $F_1^\perp$). Let $\mathcal{A}$ be any graded-commutative, non-positively graded, quasi-coherent dg-algebra on E, quasi-isomorphic to $\mathcal{O}_{F_1} \overset{L}{\otimes}_{\mathcal{O}_E} \mathcal{O}_{F_2}$ (see 1.4). Let $\pi : E \to X$ be the natural projection. Then it is well-known (see *e.g.* [Gro61a, 1.4.3]) that the functor π_* induces equivalences of categories

$$\mathcal{C}(E, \mathcal{A}) \cong \mathcal{C}(X, \pi_*\mathcal{A}), \quad \mathcal{D}(E, \mathcal{A}) \cong \mathcal{D}(X, \pi_*\mathcal{A}).$$

Moreover, the data of $\mathcal{A}$ is equivalent to the data of the $\pi_*\mathcal{O}_E$-dg-algebra $\pi_*\mathcal{A}$, which is quasi-isomorphic to $\pi_*\mathcal{O}_{F_1} \overset{L}{\otimes}_{\pi_*\mathcal{O}_E} \pi_*\mathcal{O}_{F_2}$.

Now there are natural isomorphisms $\pi_*\mathcal{O}_E \cong \mathrm{S}_{\mathcal{O}_X}(\mathcal{E}^\vee)$, $\pi_*\mathcal{O}_{F_i} \cong \mathrm{S}_{\mathcal{O}_X}(\mathcal{F}_i^\vee)$ $(i = 1, 2)$. Consider the Koszul resolution

$$\mathrm{Sym}\big(0 \to \mathcal{F}_1^\perp \to \mathcal{E}^\vee \to 0\big) \xrightarrow{\text{qis}} \mathrm{S}(\mathcal{F}_1^\vee) \cong \mathrm{S}(\mathcal{E}^\vee)/(\mathcal{F}_1^\perp \cdot \mathrm{S}(\mathcal{E}^\vee)),$$

where $\mathcal{F}_1^\perp$ is in degree -1, $\mathcal{E}^\vee$ is in degree 0, and the differential is the natural inclusion. This is a flat dg-algebra resolution of $\mathrm{S}(\mathcal{F}_1^\vee)$ over $\mathrm{S}(\mathcal{E}^\vee)$. If we tensor this resolution with $\mathrm{S}(\mathcal{F}_2^\vee)$ (over $\mathrm{S}(\mathcal{E}^\vee)$) we obtain that the dg-algebra $\mathrm{Sym}(\mathcal{X})$ is quasi-isomorphic to $\pi_*\mathcal{O}_{F_1} \overset{L}{\otimes}_{\pi_*\mathcal{O}_E} \pi_*\mathcal{O}_{F_2}$. Hence we can take $\pi_*\mathcal{A} = \mathrm{Sym}(\mathcal{X})$. This finishes the proof of the lemma. $\square$

4.2 Linear Koszul duality

One can also consider $\mathcal{X}$ as a $\mathbb{G}_{\mathbf{m}}$-equivariant $\mathcal{O}_X$-dg-module, where $\mathcal{F}_1^\perp$ and $\mathcal{F}_2^\vee$ are in internal degree 2. Then, similarly, $\mathcal{Y}$ is $\mathbb{G}_{\mathbf{m}}$-equivariant (with generators in internal degree -2). Geometrically, this corresponds to considering a $\mathbb{G}_{\mathbf{m}}$-action on E, where $t \in \Bbbk^\times$ acts by multiplication by t^{-2} along the fibers. We will use the notations

$$\begin{aligned} \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) &:= \mathcal{D}^{\mathrm{fg}}_{\mathrm{gr}}(X, \mathrm{Sym}(\mathcal{X})), \\ \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^\perp \overset{R}{\cap}_{E^*} F_2^\perp) &:= \mathcal{D}^{\mathrm{fg}}_{\mathrm{gr}}(X, \mathrm{Sym}(\mathcal{Y})). \end{aligned}$$

Then Theorem 3.7.3 gives, in this situation:

Theorem 4.2.1. *There exists a contravariant equivalence of triangulated categories, called* linear Koszul duality*,*

$$\kappa : \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) \xrightarrow{\sim} \mathcal{D}^{\mathrm{c}}_{\mathbb{G}_{\mathbf{m}}}(F_1^{\perp} \overset{R}{\cap}_{E^*} F_2^{\perp})$$

satisfying $\kappa(\mathcal{M}[n]\langle m\rangle) = \kappa(\mathcal{M})[-n+m]\langle m\rangle$.

4.3 Equivariant version of the duality

Finally, let us consider an algebraic group G acting on X (algebraically). Assume that E is a G-equivariant vector bundle, and that F_1 and F_2 are G-equivariant subbundles. Then, with the same notations as above, $\mathcal{X}$ is a complex of G-equivariant coherent sheaves on X. Let us denote by

$$\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2)$$

the derived category of $G\times\mathbb{G}_{\mathbf{m}}$-equivariant quasi-coherent $\mathrm{Sym}(\mathcal{X})$-dg-modules on X (*i.e.* $\mathbb{G}_{\mathbf{m}}$-equivariant dg-modules as above, endowed with a structure of G-equivariant quasi-coherent $\mathcal{O}_X$-module compatible with all other structures) with locally finitely generated cohomology, and similarly for $\mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^{\perp} \overset{R}{\cap}_{E^*} F_2^{\perp})$. Then our constructions easily extend to give the following result.

Theorem 4.3.1. *There exists a contravariant equivalence of categories*

$$\kappa : \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1 \overset{R}{\cap}_E F_2) \xrightarrow{\sim} \mathcal{D}^{\mathrm{c}}_{G\times\mathbb{G}_{\mathbf{m}}}(F_1^{\perp} \overset{R}{\cap}_{E^*} F_2^{\perp})$$

satisfying $\kappa(\mathcal{M}[n]\langle m\rangle) = \kappa(\mathcal{M})[-n+m]\langle m\rangle$.

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